SOLUTIONS TO EXAM 1

Problem 1 (30 pts)

(a) Since the multipliers are all 3, the row operations we had goes:

- subtract three times row 1 to row 2;
- subtract three times row 1 to row 3;
- subtract three times row 2 to row 3,

where the end step gives us

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & g \end{bmatrix}.$$

So now we just need to reverse the row operations. Reversing the last step means adding three times row 2 to row 3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 6 & g+3 \end{bmatrix}.$$

Reversing the second step means adding three times row 1 to row 3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 3 & 9 & g+6 \end{bmatrix},$$

And reversing the first step means adding three times row 1 to row 2:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 9 & g+6 \end{bmatrix}.$$

(b) To find the nullspace of A, we first note that there are three columns in A, and that the rank of A is 2 in the case when $g = 2$, so the dimension of $N(A)$ is one. So we need to find just one vector that are in $N(A)$; then this vector will be a basis for $N(A)$. By using Gaussian elimination to $Ax = 0$, we get $Ux = 0$, from which we deduce that $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ is a solution.

Therefore, $N(A) = c \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

(c) If $g \neq 0$, then we observe that $C(U) = \mathbb{R}^3$. Since row rank is equal to the column rank, the column rank (i.e. the dimension of the column space) or A is also 3. Since the column space of A is contained in $\mathbb{R}^3$, which is 3-dimensional, it must be the whole space as well. So $C(A) = \mathbb{R}^3$.

Problem 2 (40 pts)

(a) Any row of A is orthogonal to the two special solutions given in the problem. That is, any row

$$r = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

satisfies $r \cdot s_1 = r \cdot s_2 = 0$. This is just a system of two linear equations, so we need to solve the equation

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 6 & 0 & 2 & 1 \end{bmatrix} r = 0$$

whose complete solution is given by

$$c \begin{bmatrix} 1 \\ -3 \\ 0 \\ -6 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix},$$

from which we get the reduced row echelon form of A given by

$$\begin{bmatrix} 1 & -3 & 0 & -6 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

(b) R has two pivots, and therefore A has two pivots and $r(A) = 2$. Two independent columns in $\mathbb{R}^2$ span $\mathbb{R}^2$, so $C(A) = \mathbb{R}^2$.

Partial credit was given if the student referred back to A for the column space and if they gave $\mathbb{R}^2$ with incomplete reasoning. Most point were lost if they just gave $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as the basis without indicating where they came from (or from reading them off of R, not A). There were lots of right answer, with wrong (or no) reasons.

(c) The free variables are x_2, x_4 so the particular solution is

$$x_p = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix}.$$

The complete solution is

$$x_c = x_p + c_1 \cdot s_1 + c_2 \cdot s_2 = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

(d) We have

$$2 \cdot \begin{bmatrix} -3 \\ 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -6 \\ -2 \end{bmatrix}.$$

One can find these coefficients by inspection, or combining the special solutions:

$$2s_1 - s_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \\ -1 \end{bmatrix}.$$

Problem 3 (30 pts)

(a) Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a column of X. Then x, y and z satisfy

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

from which we deduce that $y = 0$, and $x = -z$. So each column of X is a multiple of the vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Since there are two columns of X, X can be written as

$$\begin{bmatrix} a & b \\ 0 & 0 \\ -a & -b \end{bmatrix}.$$

The basis for this space of matrices is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) We first solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Elimination gives We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

From which we see that we can take $y = -1/2, x = 3/2, z = 0$. This will be the first column of X.

We now solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Elimination gives We apply elimination to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And we can take $y = 1/2, x = -1/2, z = 0$. This is the second column of X.

So one possible solution for X is

$$\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \\ 0 & 0 \end{bmatrix}.$$

(c) The set of complete solutions is given by

$$X_{\text{particular}} + X_{\text{special}} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \\ 0 & 0 \end{bmatrix} + a \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$