# Spring '13 18.06 Final Solutions

1. a)

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- dim nullspace $= 1$
  $\Rightarrow$ bottom row 0s
- last entry of nullspace vector is 0, so last column should be a pivot column

b) We must have

$S$ is orthogonal to $T$ and $\dim S + \dim T = 5$

$Y$ is orthogonal to $Z$ and $\dim Y + \dim Z = 3$

Also, $\dim Y = \dim S$

$\therefore \dim T = \dim Z + 2$

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2. a) Column space: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Right nullspace: $\left\{ \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \\ -\frac{1}{4} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{3} \end{pmatrix} \right\}$

Row space: $\{(1\ 2\ 0\ 4), (0\ 0\ 1\ 3)\}$

Left nullspace: $\{(0\ 0\ 1)\}$

2. cont 

b) We have

$$A^T A = V\Sigma^T \Sigma V^T = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

So we may take $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}$.

Then $3u_1 = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ $\quad u_1 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

$3u_2 = A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ $\quad u_2 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$

$u_3$ ortho to $u_1$ and $u_2$ $\Rightarrow$ can take $u_3 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$

$$U = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

3.

a) Projection of $b$ onto column space of $A$

$$= q_1 + 2q_2 + 3q_3$$

If $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then $Ax = q_1 + 2q_2 + 3q_3$

$\therefore$ $\hat{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ best least squares approx

b) $P = q_1 q_1^T + q_2 q_2^T + q_3 q_3^T$

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4. a) $A^{-1} = A^T = A$ (symmetric)

~~(symm~~ (orthogonal)

b) Only $\pm 1$ can be eigenvalues of $A$, since it has real eigenvalues (since symmetric) with absolute value 1 (since orthogonal).

c) Trace $A$ = Sum of eigenvalues $= 2$

$\therefore$ eigenvalues must be $(1, 1, 1, -1)$

5. a)

$$Aq_1 = q_1, \quad Aq_2 = 2q_2, \quad Aq_3 = 5q_3$$

b) Write $u(0) = a_1 q_1 + a_2 q_2 + a_3 q_3$

Then $u(t) = e^{t} a_1 q_1 + e^{2t} a_2 q_2 + e^{5t} a_3 q_3$

c) Thus if $u(0) = q_1 - q_3$,

$$u(t) = e^{t} q_1 - e^{5t} q_3.$$

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6. a)

$$S = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array} & \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is a vector in $N(S)$

$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ is a vector in $N(S^T)$.

6. cont

b) $$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ is a vector in $N(A^T)$.

c) Rank $A = 9 - \dim(\text{null } A) = 8$

Dim null $(A^T) = 12 - \text{rank } A = 4$.

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7. a) $\cancel{A = \begin{pmatrix} C & B \end{pmatrix}}$ $\quad A = \begin{pmatrix} 0 & 1 \\ C & B \end{pmatrix}$

b) $\det(A - \lambda I) = 0$

that is $$\det\begin{pmatrix} -\lambda & 1 \\ C & B-\lambda \end{pmatrix} = -\lambda(B-\lambda) - C$$
$$= \lambda^2 - B\lambda - C = 0$$

c) $A\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ C & B \end{pmatrix}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ C + B\lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_1^2 \end{pmatrix}$ by the equation in b)

so $A\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$.

7. cont

d) $\lambda_1 \neq \lambda_2 \Rightarrow$ can write $u_0 = c_1 \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

So $u_n = c_1 \lambda_1^n \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 \lambda_2^n \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

and $y_n = c_1 \lambda_1^n + c_2 \lambda_2^n$.

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8.

a) $\det A = \det \begin{pmatrix} 1 & b & 0 \\ b & 1 & b \\ 0 & b & 1 \end{pmatrix} - b \det \begin{pmatrix} b & b & 0 \\ 0 & 1 & b \\ 0 & b & 1 \end{pmatrix}$

$= (1 - b^2 - b^2) - b(b - b^3)$

$= 1 - 3b^2 + b^4$

b) $\det A =$

$1 \cdot 1 \cdot 1 \cdot 1 \quad - 1 \cdot 1 \cdot b \cdot b \quad - 1 \cdot b \cdot b \cdot 1 \quad - b \cdot b \cdot 1 \cdot 1$

$+ b \cdot b \cdot b \cdot b$ all other terms $0$

$= 1 - 3b^2 + b^4$

9. a)

$$x^TAx = x^Tvv^Tx$$
$$= (v^Tx)^T(v^Tx)$$
$$= |v^Tx|^2 \geq 0.$$

b) We must have $r = n$ (i.e. null $A = 0$)
This implies $m \geq n$ since $r \leq m$ always.

Then if $x \neq 0$
$$x^TA^TAx = (Ax)^T(Ax)$$
$$= |Ax|^2 > 0 \quad \text{since null } A = 0$$

On the other hand, if $r < n$ then there is some $v \neq 0$ in null $A$. Then
$$v^TA^TAv = 0$$
so not positive definite.