18.06 Spring 2013 – Problem Set 4 Solutions

1. 8.2 #8:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

2. 8.2 #9: Row reduce the augmented matrix:

$$\left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & b_1 \\ -1 & 0 & 1 & 0 & b_2 \\ 0 & -1 & 1 & 0 & b_3 \\ 0 & -1 & 0 & 1 & b_4 \\ 0 & 0 & -1 & 1 & b_5 \end{array}\right] \to \left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & b_1 \\ 0 & -1 & 1 & 0 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 + b_1 \\ 0 & 0 & -1 & 1 & b_4 - b_2 + b_1 \\ 0 & 0 & 0 & 0 & b_5 - b_4 + b_2 - b_1 \end{array}\right]$$

   So the requirements are $b_3 - b_2 + b_1 = 0$ and $b_5 - b_4 + b_2 - b_1 = 0$. This is Kirchoff's **voltage** law around the two **loops** in the graph. (Note that the requirement for the third loop, $b_3 - b_5 + b_4 = 0$, follows from the other two. Based on the graph, can you guess why this should be the case?)

3. 8.2 #11: We have

$$A^T A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

   (a) The diagonal of $A^T A$ tells how many **edges** into each node.

   (b) The off-diagonals -1 or 0 tell which pairs of nodes are **adjacent**.

4. 4.1 #3:

   (a)

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$$

   (b) This is impossible, since the null space must be orthogonal to the row space but we have

$$\begin{bmatrix} 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = 4 \neq 0.$$

   (c) This is impossible. Since $Ax = (1,1,1)^T$ has a solution, $(1,1,1)^T$ must be in the column space $C(A)$. But $C(A)$ must be orthogonal to $N(A^T)$, and we have

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \neq 0.$$

(d)

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

(e) This is impossible, as the vector $(1, 1, \ldots, 1)^T$ would then be in both the row space and the null space which are orthogonal to each other.

4.1 #9: If $A^T Ax = 0$ then $Ax = 0$. Reason: Ax is in the nullspace of A^T and also in the **column space** of A and those spaces are **orthogonal**.

5. 4.1 #10:

   (a) Since A is symmetric, the column space is equal to the row space and is therefore orthgonal to the nullspace.

   (b) x is in the nullspace and z is in the column space, hence $x^T z = 0$.

4.1 #11: For A: The nullspace is spanned by $(-2, 1)$, the row space is spanned by $(1, 2)$. The column space is the line through $(1, 3)$ and $N(A^T)$ is the perpendicular line through $(3, -1)$. For B: The nullspace of B is spanned by $(0, 1)$, the row space is spanned by $(1, 0)$. The column space and left nullspace are the same as for A.

6. 4.1 #30: Since $N(A)$ contains $C(B)$, we have $\dim(N(A)) \geq \dim C(B)$. But $\dim(N(A)) = 4 - \text{rank}(A)$ and $\dim(C(B)) = \text{rank(B)}$. So we have $4 - \text{rank}(A) \geq \text{rank}(B)$, or $\text{rank}(A) + \text{rank}(B) \leq 4$.

7. 4.2 #3:

   (a) We have

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad Pb = \frac{1}{3}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}.$$

   One can check that $P^2 = P$.

   (b) We have

$$P = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}, \qquad Pb = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

   One can check that $P^2 = P$.

8. 4.2 #11:

   (a) We have

$$A^T A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad A^T b = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

   We then solve $A^T Ax = (5, 2)^T$ to find $x = (-1, 3)^T$. We then have $p = Ax = (2, 3, 0)^T$. We also have $e = b - p = (0, 0, 4)^T$, which is indeed orthogonal to the columns of A.

(b) We have

$$A^T A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}, \qquad A^T b = \begin{bmatrix} 14 \\ 8 \end{bmatrix}.$$

We then solve $A^T A x = (14, 8)^T$ to find $x = (-2, 6)^T$. We then have $p = Ax = (4, 4, 6)^T$. We also have $e = b - p = 0$, which is orthogonal to the columns of A. In fact we have shown that b is in the columns space of A, since $p = b$.

(c) 4.2 #31: First we need to check that p is contained in the subspace spanned by the a's. We can do this by using row reduction to determine if the system $Ax = p$ has a solution, where A is the $m \times n$ matrix with columns $a_1, \ldots, a_n$. If p is indeed in this subspace, we then compute $e = b - p$ and check that e is orthogonal to the vectors $a_1, \ldots, a_n$, i.e. $a_i^T e = 0$ for $i = 1, \ldots, n$.

```
%%%%%%%%%%%%%%%
% Problem10.m %
%%%%%%%%%%%%%%%
clear,clc
M = CompleteGraph(5)
A =  edgelist2incidence(M)
ColSpaceBasis = getcolspacebasis(A)
RowSpaceBasis = getcolspacebasis(A')
L = A'*A
N = null(L); N = N./min(N); NullSpace = rat(N)

B = ColSpaceBasis;
b = 1:10; b = b'

disp('use backslash')
x = B\b

disp('use pseudo inverse')
x = pinv(B)*b

disp('use least squares')
x = lsqlin(B,b)
x = (B'*B) \ (B'*b)

% projection onto column space
p = B*x


%% OR form Areduced and solve these equations
% remove last column (check the remaining columns still form a basis, e.g. with rref)
% in fact, Ar = B above, so we are repeating the same calculation here
Ar = A(:,1:end-1);
Lr = Ar'*Ar
xr = Lr \ (Ar'*b)
p = Ar*xr

%%%%%%%%%%%%%%%%%%%%
% CompleteGraph.m  %
%%%%%%%%%%%%%%%%%%%%

function  M = CompleteGraph(N)
% generate edge list for complete graph on N nodes
e = 0;
for i = 1:N
    for j = i+1:N
        e = e + 1;
        M(e,1) = i;
        M(e,2) = j;
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%
% edgelist2incidence.m %
%%%%%%%%%%%%%%%%%%%%%%%%

function A = edgelist2incidence(M)
% M is m x 2 matrix
% if edge e connects node i to node j then
% row e = [i   j]
% Two conventions:
% 1. columns are increasing
% 2. i < j

m = size(M,1);
n = max(max(M));
A = zeros(m,n);
for e = 1:m
    i = M(e,1); j = M(e,2);
    A(e, i) = -1 ; A(e, j) = +1;
end
```

```
%%%%%%%%%%%%%%%%%%%%%%
% getcolspacebasis.m %
%%%%%%%%%%%%%%%%%%%%%%

function B = getcolspacebasis(A)
[E jb] = rref(A);
B = A(:,jb);
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