\[ A_1 = \begin{pmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{pmatrix}, \det(A_1 - \lambda I) = \det \begin{pmatrix} 0.6 - \lambda & 0.9 \\ 0.4 & 0.1 - \lambda \end{pmatrix} = (0.6 - \lambda)(0.1 - \lambda) - (0.4)(0.9) = \lambda^2 - 0.7\lambda - 0.3 = (\lambda - 1)(\lambda + 0.3) \]

\[ \Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix} \]

\[
\begin{pmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow 4v_1 = 9v_2 \\
\begin{pmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -0.3 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow v_1 = -v_2
\]

\[ \lambda_1 = 1, v_1 = \begin{pmatrix} 9 \\ 4 \end{pmatrix}; \lambda_2 = -0.3, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

\[ S_1 = \begin{pmatrix} 9 & 1 \\ 4 & -1 \end{pmatrix} \]

\[ A_1^k = S_1 \Lambda_1^k S_1^{-1} \rightarrow \begin{pmatrix} 9 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 9 & 1 \\ 4 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 9 & 0 \\ 4 & 0 \end{pmatrix} \frac{1}{13} \begin{pmatrix} 1 & 1 \\ 4 & -9 \end{pmatrix} \]

\[ = \frac{1}{13} \begin{pmatrix} 9 & 9 \\ 4 & 4 \end{pmatrix} \] (In the columns, we see the eigenvector \( v_1 \) for the eigenvalue \( 1 \).)
\[ A_2 = \begin{pmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{pmatrix}, \det(A_2 - \lambda I) = \det \begin{pmatrix} 0.6 - \lambda & 0.9 \\ 0.1 & 0.6 - \lambda \end{pmatrix} = (0.6 - \lambda)^2 - (0.1)(0.9) = \lambda^2 - 1.2\lambda + 0.27 = (\lambda - 0.3)(\lambda - 0.9) \]

\[ A_2 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.9 \end{pmatrix} \]

\[ \begin{pmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.3 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow v_1 = -3v_2 \]

\[ \begin{pmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.9 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow v_1 = 3v_2 \]

\[ \lambda_1 = 0.3, v_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}; \lambda_2 = 0.9, v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \]

\[ S_2 = \begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix} \]

\[ A_2^{10} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{6} S_2 A_2^{10} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 4(0.9)^{10} \end{pmatrix} = \begin{pmatrix} 2(0.9)^{10} \\ \frac{2}{3}(0.9)^{10} \end{pmatrix} \]

\[ A_2^{10} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{1}{6} S_2 A_2^{10} \begin{pmatrix} -6 \\ 0 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -6(0.3)^{10} \\ 0 \end{pmatrix} = \begin{pmatrix} 3(0.3)^{10} \\ -(0.3)^{10} \end{pmatrix} \]

\[ A_2^{10} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = A_2^{10} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + A_2^{10} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2(0.9)^{10} + 3(0.3)^{10} \\ \frac{2}{3}(0.9)^{10} - (0.3)^{10} \end{pmatrix} \]

20

\[ A = SAS^{-1}, \text{ so } \det A = (\det S)(\det \Lambda)(\det S^{-1}) = (\det \Lambda)(\det S)(\det S^{-1}) = \det \Lambda \]

This works when \( A \) can be diagonalized.

21

\[ ST = \begin{pmatrix} aq + bs & ar + bt \\ cq + ds & cr + dt \end{pmatrix} \]

\[ TS = \begin{pmatrix} qa + rc & qb + rd \\ sa + tc & sb + td \end{pmatrix} \]

\[ \text{tr}(ST) = \text{tr}(TS) = aq + bs + cr + dt \]

24 If \( S^{-1}AS \) and \( S^{-1}BS \) are diagonal, \( S^{-1}(A + B)S \) is diagonal and \( S^{-1}(cA)S \) are also diagonal. Thus, all such \( 4 \times 4 \) matrices form a subspace.

If \( S = I \), we are looking at all matrices \( A \) such that \( S^{-1}AS = A \) is diagonal; so it is the subspace of all diagonal matrices (this has dimension 4).
\[
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

\[
v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

This matrix only has 1 eigenvectors; its column space and nullspace are both spanned by \(v\) (so they coincide).

35

\[
B - \lambda I = \begin{pmatrix} 3 - \lambda & 2 \\ -5 & -3 - \lambda \end{pmatrix}
\]

\[
C - \lambda I = \begin{pmatrix} 5 - \lambda & 7 \\ -3 & -4 - \lambda \end{pmatrix}
\]

\[
\det(B - \lambda I) = (3 - \lambda)(-3 - \lambda) - 2(-5) = \lambda^2 + 1
\]

\[
\det(C - \lambda I) = (5 - \lambda)(-4 - \lambda) - 7(-3) = \lambda^2 - \lambda + 1
\]

Let the eigenvalues of \(B\) (resp. \(C\)) be \(b_1, b_2 (c_1, c_2)\); then \(b_1^4 = b_2^4 = 1, c_1^3 = -1, c_2^3 = -1\); so the eigenvalues of \(B^4\) (resp. \(C^3\)) are 1, 1 (resp. \(-1, -1\)). So \(B^4 = I, C^3 = -I\).

6.3

1

\[
A = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}
\]

\[
A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 4v_1 + 3v_2 \\ v_2 \end{pmatrix}
\]

The eigenvalues of \(A\) are \(\lambda_1 = 4, \lambda_2 = 1\) (because \(A\) is upper triangular); the eigenvectors are:

\[
v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

Solutions:

\[
u = c_1 e^{4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 e^{4t} + c_2 e^t \\ -c_2 e^t \end{pmatrix}
\]

\[
t = 0 \Rightarrow c_1 + c_2 = 5, -c_2 = -2 \Rightarrow c_1 = 3, c_2 = 2, u = \begin{pmatrix} 3e^{4t} + 2e^t \\ -2e^t \end{pmatrix}
\]
4

\[ A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \]

\[ \det(A - \lambda I) = \det \begin{pmatrix} -1 - \lambda & 1 \\ 1 & -1 - \lambda \end{pmatrix} = (-1 - \lambda)^2 - 1 = \lambda(\lambda + 2) = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -2 \]

\[ A \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} w - v \\ v - w \end{pmatrix} = \begin{pmatrix} \lambda v \\ \lambda w \end{pmatrix}; \lambda = \lambda_1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \lambda = \lambda_2 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

Solutions: \[ u = c_1 e^{0t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \end{pmatrix} \]

\[ t = 0 \Rightarrow \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 30 \\ 10 \end{pmatrix} \]
\[ c_1 = 20, c_2 = 10 \Rightarrow \begin{pmatrix} 20 + 10e^{2t} \\ 20 - 10e^{2t} \end{pmatrix} \]

14 (a)

\[ u_1 u_1' + u_2 u_2' + u_3 u_3' = u_1(cu_2 - bu_3) + u_2(au_3 - cu_1) + u_3(bu_1 - au_2) = 0 \]

\[ \Rightarrow \frac{d}{dt}(||u(t)||^2) = 2(u_1 u_1' + u_2 u_2' + u_3 u_3') = 0 \Rightarrow ||u(t)||^2 = ||u(0)||^2 \]

(b)

\[ Q^T = (1 + At + \frac{A^2 t^2}{2!} + \cdots)^T = 1 + (A^T)t + \frac{(A^T)^2 t^2}{2!} + \cdots \]

\[ = 1 - At + \frac{A^2 t^2}{2!} + \cdots = e^{-At} \]

\[ Q^T Q = e^{-At}e^{At} = I \]

16

\[ \frac{du}{dt} = Au - e^{ct}b; u = e^{ct}v \]
\[ \frac{du}{dt} = ce^{ct}v + e^{ct}\frac{dv}{dt} = e^{ct}(cv + \frac{dv}{dt}); Au - e^{ct}b = e^{ct}(Av - b) \]

\[ \Rightarrow cv + \frac{dv}{dt} = Av - b \]
\[ \frac{dv}{dt} = (A - cI)v - b \]

If \( c \) is not an eigenvalue, \( A - cI \) is invertible; using the result from Q15, \( v = (A - cI)^{-1}b \); and

\[ u = e^{ct}(A - cI)^{-1}b \]

is a particular solution.
\[
A = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}, \quad A^2 = A
\]
\[
B = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}, \quad B^2 = 0
\]
\[
C = A + B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C^2 = C
\]
\[
e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = I + A\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) = I + A(e - 1) = \begin{pmatrix} e & 4(e - 1) \\ 0 & 1 \end{pmatrix}
\]
\[
e^B = I + B + \frac{B^2}{2!} + \cdots = I + B = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}, \quad e^C = I + C + \frac{C^2}{2!} + \cdots = I + C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
\[
e^Ae^B = \begin{pmatrix} e & -4 \\ 0 & 1 \end{pmatrix}, \quad e^Be^A = \begin{pmatrix} e & 4e - 8 \\ 0 & 1 \end{pmatrix}, \quad e^{A+B} = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}
\]

25

\[
A^2 = A
\]
\[
e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = I + A\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = I + A(e^t - 1)
\]
\[
= \begin{pmatrix} e^t & 3e^t - 3 \\ 0 & 1 \end{pmatrix}
\]

8.3

2

\[
A = \begin{pmatrix} .9 & .15 \\ .1 & .85 \end{pmatrix}
\]
\[
\lambda_1 + \lambda_2 = \text{tr}(A) = .9 + .85; \quad \lambda_1 = 1 \Rightarrow \lambda_2 = 0.75
\]
\[
\begin{pmatrix} .9 & .15 \\ .1 & .85 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0.9v_1 + 0.15v_2 \\ 0.1v_1 + 0.85v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
\]
\[
\lambda = \lambda_1 \Rightarrow v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}; \quad \lambda = \lambda_2 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]
\[
\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}, \quad S = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}, \quad S^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}
\]
\[
A^k = S\Lambda^kS^{-1} \rightarrow S \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S^{-1} = \frac{1}{5} \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}
\]
Challenge problem:

\[
v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}; \quad Av = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ (1-a)v_1 + (1-b)v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}
\]

\[
\lambda_1 + \lambda_2 = a + 1 - b; \quad \lambda_1 = 1 \Rightarrow \lambda_2 = a - b
\]

\[
\lambda = \lambda_1 = 1 \Rightarrow v_1 = \begin{pmatrix} b \\ 1-a \end{pmatrix}; \quad \lambda = \lambda_2 = a - b \Rightarrow v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

\[
\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & a - b \end{pmatrix}; \quad S = \begin{pmatrix} b & 1 \\ 1-a & -1 \end{pmatrix}; \quad S^{-1} = \frac{1}{a-1-b} \begin{pmatrix} -1 & -1 \\ a - 1 & b \end{pmatrix}
\]

\[
A^k = S \Lambda^k S^{-1} \rightarrow \begin{pmatrix} b & 1 \\ 1-a & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{a-1-b} \begin{pmatrix} -1 & -1 \\ a - 1 & b \end{pmatrix} = \begin{pmatrix} \frac{-b}{a-1-b} & \frac{-b}{a-1-b} \\ \frac{a-1-b}{a-1-b} & \frac{a-1-b}{a-1-b} \end{pmatrix}
\]

\[
\begin{pmatrix} \frac{-b}{a-1-b} & \frac{-b}{a-1-b} \\ \frac{a-1-b}{a-1-b} & \frac{a-1-b}{a-1-b} \end{pmatrix} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix} \Leftrightarrow \frac{-b}{a-1-b} = 0.6 \Leftrightarrow 0.4b + 0.6a = 0.6
\]

For \( a = 0.8, b = 0.3 \) this is true.

\[
A^2 = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}^2 = \begin{pmatrix} a^2 + b - ab & ab + b - b^2 \\ -a^2 + 1 + ab - b & -ab + 1 + b^2 - b \end{pmatrix}
\]

All the entries of \( A^2 \) are positive (since all entries of \( A \) are positive); and the columns of this matrix also add up to 1, so \( A^2 \) is Markov.

\[
(I + A)(I - A + A^2 - A^3 + \cdots) = (I - A + A^2 - A^3 + \cdots) + (A - A^2 + A^3 + \cdots) = I
\]

\[
A^2 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}; \quad A^{2k} = \begin{pmatrix} 1/2^k & 0 \\ 0 & 1/2^k \end{pmatrix}, \quad A^{2k+1} = A \cdot A^{2k} = \begin{pmatrix} 0 & 2^{k+1} \\ 1/2^k & 0 \end{pmatrix}
\]

\[
I + A + A^2 + \cdots = \begin{pmatrix} 1 + 1/2 + 1/4 + \cdots \\ 1 + 1/2 + 1/4 + \cdots \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}
\]