1. (15 points)

(a) If A is a 3 by 4 matrix, what does this tell us about its nullspace?

Solution: $dim\, N(A) \geq 1$, since $rank(A) \leq 3$.

(b) If we also know that

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

has no solution, what do we know about the rank of A?

Solution: $C(A)$ does not span the entire $\mathbf{R}^3$, so $rank(A) \leq 2$.

(c) If $Ax = b$ and $A^T y = 0$, find $y^T b$ by using those equations. This says that the _____ space of A and the ________________________ are __________.

Solution: $y^T b = y^T (Ax) = (A^T y) x = 0$. This says that the **column** space of A and the **null space** of A^T are **orthogonal**.

2. (15 points) Suppose $Ax = b$ reduces by the usual row operations to $Ux = c$:

$$Ux = \begin{bmatrix} 2 & 6 & 4 & 8 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 - 2b_2 + b_1 \end{bmatrix} = c.$$

(a) Give a basis for the nullspace of A (that matrix is not shown) and a basis for the row space of A.

Solution: $N(A) = N(U)$, $R(A) = R(U)$. Therefore we can read the bases directly from U:

$$N(A) = span \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$R(A) = span \left\{ \begin{bmatrix} 2 \\ 6 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 4 \end{bmatrix} \right\}$$

(b) When does $Ax = b$ have a solution? Give a basis for the column space of A.

Solution: $b \in C(A)$, equivalent to $c \in C(U)$.

By looking at c as a function of b we can reconstruct A. Let $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$. We have $U = EA$, $c = Eb$. Hence $A = E^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 & 8 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 & 8 \\ 2 & 6 & 8 & 12 \\ 2 & 6 & 12 & 16 \end{bmatrix}$

From here we see that

$$C(A) = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

(c) Give a basis for the nullspace of A^T.

Solution: $N(A^T) \perp C(A)$. So $N(A^T) = span \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ (we only need to find a vector orthogonal to both basis vectors we gave for $C(A)$).

3. (10 points)

(a) Suppose q_1, q_2 are orthonormal in $\mathbb{R}^4$, and v is NOT a combination of q_1 and q_2. Find a vector q_3 by Gram-Schmidt, so that q_1, q_2, q_3 is an orthonormal basis for the space spanned by q_1, q_2, v.

Solution:

$$q_3 = \frac{v - (v^T q_1)q_1 - (v^T q_2)q_2}{\|v - (v^T q_1)q_1 - (v^T q_2)q_2\|}$$

(b) If p is the projection of b onto the subspace spanned by q_1 and q_2 and v, find p as a combination of q_1, q_2, q_3. (You are solving the least squares problem $Ax = b$ with $A = [q_1, q_2, q_3]$.)

Solution:

$$p = A(A^T A)^{-1} A^T b = AA^T b = q_1(q_1^T b) + q_2(q_2^T b) + q_3(q_3^T b)$$

where we used the fact that the columns of A are orthonormal for $A^T A = I$. (It is easy to see the final result just by thinking that we are actually projecting onto an orthonormal basis).

4. (10 points)

(a) To solve a square system $Ax = b$ when det $A \neq 0$, Cramer's Rule says that the first component of x is

$$x_1 = \frac{\det B}{\det A} \quad \text{with} \quad B = [b\, a_2 \dots a_n]\,.$$

So b goes into the first column of A, replacing a_1. If $b = a_1$, this formula gives the right answer $x_1 = \frac{\det A}{\det A} = 1$.

1. If $b =$ a different column a_j, show that this formula gives the right answer, $x_1 =$ _____.
2. If b is any combination $x_1 a_1 + \cdots + x_n a_n$, why does this formula give the right answer x_1?

Solution:

1. $x_1 = 0$, since $\det(B) = 0$. Let us check that it gives the correct answer: the solution to $Ax = a_j$ is $x = e_j$ (it is unique, since $\det(A) \neq 0$, so A's columns are linearly independent). Hence $x_1 = 0$.
2. For the same reason as above the first component of x is indeed x_1, the coefficient of a_1.

   Now let us see what Cramer's Rule gives. In this case $\det(B) = \det([b\, a_2 \, \dots a_n]) = \det(x_1 \cdot a_1\, a_2 \, \dots a_n) = x_1 \det(A)$. So, indeed, $x_1 = \det(B)/\det(A)$.

(b) Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 0 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{bmatrix}.$$

Cofactor expansion by the first row:

$$C_{11} = \det \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 7 \\ 0 & 8 & 9 \end{bmatrix} = 4 \cdot 6 \cdot 9 - 4 \cdot 8 \cdot 7 = -8$$

$$C_{12} = -\det \begin{bmatrix} 3 & 0 & 0 \\ 5 & 6 & 7 \\ 0 & 8 & 9 \end{bmatrix} = -(3 \cdot 6 \cdot 9 - 3 \cdot 7 \cdot 8) = 6$$

So $\det(A) = 1 \cdot (-8) + 2 \cdot 6 = 4$

5. (15 points)

(a) Suppose an n by n matrix A has n independent eigenvectors $x_1, \ldots, x_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. What matrix equation would you solve for $c_1, \ldots, c_n$ to write the vector u_0 as a combination $u_0 = c_1x_1 + \cdots + c_nx_n$?

Solution:

$$[x_1\, x_2 \ldots x_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = u_0$$

(b) Suppose a sequence of vectors $u_0, u_1, u_2, \ldots$ starts from u_0 and satisfies $u_{k+1} = Au_k$. Find the vector u_k as a combination of $x_1, \ldots, x_n$.

Solution: Let $u_0 = c_1x_1 + \cdots + c_nx_n$. Then $u_k = A^k u_0 = \sum_{i=1}^{n} (\lambda_i^k c_i) x_i$.

(c) State the exact requirement on the eigenvalues λ so that $A^k u_0 \to 0$ as $k \to \infty$ for every vector u_0. Prove that your condition **must hold**.

Solution: $|\lambda_i| < 1$, for all i. Clearly, if this holds, all the coefficients of x_i in $A^k u_0$ go to 0 as $k \to \infty$.

For the converse, we require that the coefficient $\lambda_i^k c_i$ of x_i in $A^k u_0$ to go to 0, for any choice of u_0. Equivalently, we need $\lambda_i^k c_i \to 0$ for any c_i. Hence $|\lambda_i| < 1$.

6. (10 points)

(a) Find the eigenvalues of this matrix A (the numbers in each column add to zero).

$$A = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ 1 & -1 & 1 \\ 0 & \frac{1}{2} & -1 \end{bmatrix}.$$

Solution: The number in each column add to zero, hence $\mathbf{1} \in N(A^T)$, so $\dim N(A^T) > 0$, and thus $rank(A) = rank(A^T) < 3$. So $\lambda_1 = 0$.

We can easily spot $\lambda_2 = -1$ as another eigenvalue, since subtracting $A + I$ has two equal columns, and hence $\det(A + I) = 0$.

Looking at the trace we get that $\lambda_3 = tr(A) - \lambda_1 - \lambda_2 = -2$.

(b) If you solve $\frac{du}{dt} = Au$, is (1) or (2) or (3) true as $t \to \infty$?

(1) $u(t)$ goes to zero?

(2) $u(t)$ approaches a multiple of (what vector?)

(3) $u(t)$ blows up?

Solution: Approaches a multiple of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Observe that $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector for the 0 eigenvalue, and that the only nonzero eigenvalue of e^{At} as $t \to \infty$ is $e^{0 \cdot t} = 1$, with the same eigenvector.

Also, $u(t) = e^{At}u(0)$, and that the only nonzero eigenvalue of e^{At} (as $t \to \infty$) is 1, with the same eigenvector. So $\lim_{t \to \infty} u(t)$ is a projection of $u(0)$ on the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, hence a multiple of it.

7. (10 points) Every invertible matrix A equals an orthogonal matrix Q times a positive definite matrix S. This famous fact comes directly from the SVD for the square matrix $A = U\Sigma V^T$, by choosing $Q = UV^T$.

(a) How can you prove that $Q = UV^T$ is orthogonal?

Solution: Q is orthogonal if and only if $Q^TQ = I$. Notice that since A is invertible, U and V are both square matrices of full rank.

$$Q^TQ = (UV^T)^T(UV^T) = VU^TUV^T = V(U^TU)V^T = VV^T = I$$

We used the fact that U is orthogonal, hence $U^TU = I$, and that V^T is orthogonal because V's columns are eigenvectors of a symmetric matrix A^TA, so $V^T = V^{-1}$.

(b) Substitute Q^{-1} and A to write $S = Q^{-1}A$ in terms of U, V and Σ. How can you tell that this matrix S is symmetric positive definite?

Solution:

$$S = Q^{-1}A = (UV^T)^{-1}A = (V^T)^{-1}U^{-1}U\Sigma V^T = V\Sigma V^T = (\Sigma^{1/2}V^T)^T(\Sigma^{1/2}V^T)$$

8. (15 points) A 4-node graph has all six possible edges. Its incidence matrix A and its Laplacian matrix $A^T A$ are

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

(a) Describe the nullspace of A.

Solution: $N(A) = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (Recall that applying A to a vector of potentials gives the potential drops along edges, so in order for a vector of potentials to be in the null space, all the potentials within one connected component must be the same.)

(b) The all-ones matrix $B = \text{ones}\,(4)$ has what eigenvalues? Then what are the eigenvalues of $A^T A = 4I - B$?

Solution: $B = \vec{1} \cdot \vec{1}^T = 4(\vec{1}/2)(\vec{1}/2)^T$, where $\vec{1}$ is the all-ones vector in $\mathbb{R}^4$. So B has eigenvalues $4, 0, 0, 0$.

I and B diagonalize in the same eigenbasis, so $\lambda_i(4I - B) = \lambda_i(4I) - \lambda_i(B) = 4\lambda_i(I) - \lambda_i(B)$ for all i. So the eigenvalues of $A^T A$ are $0, 4, 4, 4$.

(c) For the Singular Value Decomposition $A = U\Sigma V^T$, can you find the nonzero entries in the diagonal matrix Σ and one column of the orthogonal matrix V?

Solution: $\sigma_i = \sqrt{\lambda_i(A^T A)}$, so the nonzero singular values are $2, 2, 2$. We only need to find one eigenvector of $A^T A$. An obvious one is $\vec{1}/2$, since all the rows sum up to 0.