Problem Set # 1 Solution, 18.06

For grading: Each problem worths 10 points, and there is 5 points of extra credit in problem 8. The total maximum is 100.

1. (10pts) In Lecture 1, Prof. Strang drew the cone (infinite triangle) that comes from all combinations $cv + dw$ with $c \geq 0$ and $d \geq 0$. Which c and d would give that triangle cut off by a top line from v to w? Which c and d give the parallelogram that starts with sides v and w ?

   Solution: The points on the line connecting v and w, by vector addition, is represented by

   $$w + t(v - w) = (1 - t)w + tv, \texttt{with } 0 \leq t \leq 1$$

   therefore the top boundary is characterized by $c + d = 1, 0 \leq c, 0 \leq d$. Together with the other boundary (v and w), the triangle is given by

   $$c + d \leq 1, c \geq 0, d \geq 0.$$

   For the parallelogram, each point could be reached by starting from cv with $0 \leq c \leq 1$ then go the other direction dw with $0 \leq d \leq 1$. Therefore the answer is

   $$0 \leq c \leq 1, 0 \leq d \leq 1.$$

2. (10pts) The length of v is $||v|| = \sqrt{v' * v} = \sqrt{v_1^2 + ... + v_n^2}$. The dot product $v' * w$ equals $||v||||w||$ times the cosine of the angle between v and w. If $||v|| = 3$ and $||w|| = 5$, what are the smallest and largest possible values of the dot product $v' * w$ and of $||v - w||$?

   Solution: From definition, $v' * w = \|v\|\|w\| \cos\theta = 15 \cos\theta$, where θ is the angle between v and w. Since $\cos\theta$ ranges between -1 and 1, the smallest value of $v' * w$ is -15, largest is 15. The smallest value is achieved when v and w are parallel and pointing to opposite direction, while the largest is when they are parallel and pointing to the same direction.

   For $v - w$, we can use the triangle inequality (side 1 $\leq$ side 2 + side 3):

   $$|\|v\| - \|w\|| \leq \|v - w\| \leq \|v\| + \|w\|$$

   which gives

   $$2 \leq \|v - w\| \leq 8.$$

Alternatively we can also use the definition of dot product:

$\|v-w\|^2 = (v-w)\cdot(v-w) = v\cdot v-2v\cdot w+w\cdot w = \|v\|^2-2v\cdot w+\|w\|^2 = 9-2v\cdot w+25$

From the discussion above we know $-15 \leq v \cdot w \leq 15$, which gives us

$$4 \leq \|v - w\|^2 \leq 64$$

that is

$$2 \leq \|v - w\| \leq 8.$$

3. (10pts) The column vectors $u = (1, 1, 2)$, $v = (1, 2, 3)$ and $w = (3, 5, 8)$ are in a plane because w is what combination of u and v? Find two combinations of u, v, w that produce $b = (0, 0, 0)$ and two combinations that produce $b = (1, -1, c)$. What is the only possible number c that gives a vector on the plane?

   Solution: Solve $xu + yv = w$ to get

   $$w = u + 2v.$$

   From above we know

   $$u + 2v - w = (0, 0, 0)$$

   and multiplication by any constant gives the same result, for example

   $$2u + 4v - 2w = (0, 0, 0)$$

   (In fact, multiples of $(1, 2, -1)$ are th only solutions we have.)

   To get the combination for $b = (1, -1, c)$, by observation we get

   $$3u - 2v + 0w = (1, -1, 0)$$

   adding a copy of $u + 2v - w = (0, 0, 0)$ we get another copy

   $$4u + 0v - w = (1, -1, 0)$$

   (And $(3, -2, 0) + k(1, 2, -1)$ are all the solutions.)

   $c = 0$ is the only possible number to make $(1, -1, c)$ lie on the plane.

   These problems come from Introduction to Linear Algebra (4th edition)

4. (10pts) Problem 19 on page 42

   Solution:

   $$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

   $$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let $x = (3, 4, 5)$

$$Ex = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$$

then multiplying by E^{-1}

$$E^{-1}(Ex) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = x$$

5. (10pts) Problem 35 on page 44

   Solution: For a Sudoku matrix S, and $x = (1, 1, ..., 1)$, Sx is a column vector with 9 elements, all equal to 45:

   $$Sx = (45, 45, ...45)$$

   There are 6 permutations of three numbers: (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1) as mentioned in Section 2.7. Group each three rows, i.e. row 1-3, 4-6 and 7-9, and we can do row permutations inside each group, which would still give us Sudoku matrix. This in total gives $6 * 6 * 6$ ways of creating new matrices.

   And exchange the order of the three row blocks will also give us Sudoku matrix. This gives another 6 ways of permutation. Combined with the row permutation inside each group, in total, we have $6^4 = 1296$ orders of 9 rows that stay Sudoku.

6. (10pts) Problem 8 on page 52

   Solution: When $k = 3$ we have $3x + 3y = 6$ and $3x + 3y = -6$, so after first step of elimination we'll have $3x + 3y = 6$ and $0 = -12$. Here elimination breaks down and we have 0 solutions.

   If $k = -3$ we have $-3x + 3y = 6$ and $3x - 3y = -6$, so after first step we'll have $-3x + 3y = 6$ and $0 = 0$. Here we have infinite number of solutions.

   When $k = 0$, we need to do a row exchange before elimination, and it gives one solution.

7. (10pts) Problem 11 on page 53

   Solution: (a) We could easily see that $\frac{1}{2}(x + X, y + Y, z + Z)$ is also a solution. In fact, any combination of $c(x, y, z) + d(X, Y, Z)$ with $c + d$ is a solution.

   (b) The 25 planes also meet on the line containing the two points.

8. (10pts + Extra credit 5pts) Problem 26 on page 54 (matrices with given row and column sums)

Solution: Adding the two equations of the 1st column gives $a+b+c+d = 12$, then subtract $a + c = 2$, we have

$$s = b + d = 10.$$

Two examples of matrices with these sums

$$\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$$

Extra credit: the system $Ax = b$ is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 2 \\ 10 \end{bmatrix}$$

Making A triangular:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we can see that the elimination breaks down at the last step. There is a solution only if $d = a + c - b$. The four columns of A lie in a 3D hyperplane.

9. (10pts) Problem 32 on page 55

   Solution:

   (a) Some linear combination of the 100 rows is the row of 100 zeros.

   (b) Some linear combination of the 100 columns is the column of zeros.

   (c) A very singular matrix has all ones. A better example has 99 random rows. The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

   (d) The row picture has 100 planes meeting along a common line through 0. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

10. (10pts) Problem 29 on page 66

    Solution:

    To eliminate column 1, we multiply by the following matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

The diagonal entries of E are all 1's, and the only other nonzero entries are the 1st column. Here row 1 is subtracted from all other rows.

Then we reduce the second column to all 1's by the following matrix (considering block operations that only changes the bottom right 3 by 3 submatrix)

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

And third column to

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

All together, multiply E_3, E_2 and E_1 we get matrix E:

$$E = E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

So what E really does is that each row is subtracted from the next row.

Similarly, for the smaller Pascal matrix, we use the following matrix to eliminate the second column:

$$E' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Then the third column

$$E'' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

All together we get M with $MA = I$. Then M must be the inverse of the Pascal matrix.

$$M = E''E'E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$