18.06 (Spring 14) Problem Set 3

This problem set is due Thursday, Feb 27, 2014 by 4pm in E17-131. The problems are out of the 4th edition of the textbook. This homework has 8 questions worth 80 points in total. Please WRITE NEATLY. You may discuss with others (and your TA), but you must turn in your own writing.

1. Explain why every vector in the column space of **AB** is also in the column space of **A**. (This will tell us an important fact: $rank(\mathbf{AB}) \leq rank(\mathbf{A})$)

   Solution: By matrix multiplication, every column vector of **AB** is a linear combination of column vectors of **A**. Therefore, every vector in the column space of **AB** is also in the column space of **A**.

2. Explain why every vector in the nullspace of **B** is also in the nullspace of **AB**. Is this also true for every vector in the nullspace of **A** or is there an example where it's not true?

   Solution: Suppose a vector **x** is in the nullspace of **B**, then we get $\mathbf{Bx} = 0$. By matrix multiplication, $\mathbf{ABx} = 0$, therefore **x** is also in the nullspace of **AB**.
   This is not true for every vector in the nullspace of **A**. Take **A**=[1,0;0,0], **B**=[1,1;0,0], and **x**=(0,1). Then $\mathbf{Ax} = 0$, but **x** is not in the nullspace of **AB**.

3. Suppose you have applied elimination to **A** and reached $\mathbf{R} = rref(\mathbf{A})$. How would you be able to describe vectors (from looking at **R**) that span the column space of **A**?
   Solution: Since **A** and its reduced form **R** have same list pivcol, we can easily get the pivot columns of **A**. The column space of **A** is spanned by those pivot column vectors.

4. Suppose columns 2 and 4 of a 5 by 5 matrix **A** are the same. Then___ is a free variable. Find the special solution that goes with this free variable.

   Solution: 4 is a free variable. (0,-1,0,1,0) is the special solution that goes with this free variable.

5. From looking at $rref(\mathbf{A})$, how can you read off all special solutions to $\mathbf{Ax} = \mathbf{0}$?

   Solution: Suppose **A** is an m by n matrix and let $\mathbf{R} = rref(\mathbf{A})$. Assume **R** has r pivot columns and $n - r$ free variables. Each special solution has one free variable equal to 1 and the other free variables are all zero. By setting one free variable equal to 1 and all others equal to 0, we get the values of pivot variables from equation $\mathbf{Rx} = 0$.

   In general, suppose the first r columns of **R** are the pivot columns, then,

   $$\mathbf{R} = \begin{bmatrix} \mathbf{I} & \mathbf{F} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

   Then the nullspace matrix **N** is given by $\mathbf{N} = \begin{bmatrix} -\mathbf{F} \\ \mathbf{I} \end{bmatrix}$

6. If **A** is 3 by 4 and **B** is 4 by 3, explain why **BA** is not the identity matrix.

Solution: Since **B** is a 4 by 3 matrix, then $rank(\mathbf{B}) \leq 3$. By Problem 1, we have $rank(\mathbf{BA}) \leq rank(\mathbf{B}) \leq 3$. Note that a 4 by 4 identity matrix has $rank = 4$, so we conclude that **BA** can not be the identity matrix.

7. Problem 1, page 163 (section 3.4). Execute the six steps of Work Example 3.4 A to describe the column space and nullspace of **A** and the complete solution to $\mathbf{Ax} = \mathbf{b}$:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

Solution:

(a) $$\begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & -1 & -1 & -2 & b_3 - b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix}$$

(b) $\mathbf{Ax} = \mathbf{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$.

(c) The column space contains all combinations of (2,2,2) and (4,5,3) or the column space contains all vectors with $b_3 + b_2 - 2b_1 = 0$.

(d) The nullspace contains all combinations of $s_1 = (-1, -1, 1, 0)$ and $s_2 = (2, -2, 0, 1)$

(e) $x_p = (4, -1, 0, 0)$ gives a particular solution. The complete solution to $\mathbf{Ax} = (4, 3, 5)$ is $\mathbf{x} = x_p + c_1 s_1 + c_2 s_2$ for all $c_1, c_2 \in \mathbb{R}$.

(f) $\begin{bmatrix} R & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ gives a particular solution $x_p = (4, -1, 0, 0)$.

8. Problem 46, page 183 (section 3.5). Suppose **A** is 10 by 10 and $\mathbf{A^2} = 0$(zero matrix). This means that the column space of **A** is contained in the ___. If **A** has rank r, those subspaces have dimension $r \leq 10 - r$. So the rank is $r \leq 5$.

Solution: If $\mathbf{A^2}$ is a zero matrix, this means that column space of **A** is contained in the nullspace of **A**.

9. (Not to turn in) PLEASE practice finding $\mathbf{R} = rref(\mathbf{A})$ and also $\mathbf{N}(\mathbf{A})$—choose a 3 by 5 matrix **A** with $rank$ 2.