18.06 (Spring 14) Problem Set 4

This problem set is due Thursday, Mar. 13, 2014 by 4pm in E17-131. The problems are out of the 4th edition of the textbook. This homework has 8 questions worth 100 points in total. Please WRITE NEATLY. You may discuss with others (and your TA), but you must turn in your own writing.

1. How far is the point $b = (1, 2, 2, 4)$ from the line through the origin in the direction of the vector $a = (1, 7, 7, 1)$?

   solution

   Consider the projection of b onto this line, let's call it $b_{\|}$. We have

   $$|b_{\|}| = \frac{a \cdot b}{|a|^2}|a| = \frac{33}{100}10 = 3.3$$

   From the Pythagorean theorem, the distance from the point b to the line is given by

   $$\sqrt{|b|^2 - |b_{\|}|^2} = \sqrt{14.11}$$

2. How far is that point b from the line that goes through $c = (4, 1, 2, 2)$ in the direction of the same vector a ?

   solution

   We know that $b-c$ is a vector from the line to the given point. Consider the projection of $b - c$ onto this direction, repeat every step in the previous solution. We have

   $$b - c = (-3, 1, 0, 2)$$
   $$|(b - c)_{\|}| = 0.6\sqrt{|b - c|^2 - |(b - c)_{\|}|^2} = \sqrt{13.64}$$

   Therefore, the distance is $\sqrt{13.64}$.

3. How far is the point $b = (1, 2, 2)$ from the plane $3x + 2y + 6z = 0$?

   solution

   The normal direction of the plane is given by $n = (3, 2, 6)$. The origin is in the plane. We can project b onto the normal direction to get

   $$\frac{b \cdot n}{|n^2|}n = \frac{19}{49}(3, 2, 6).$$

   Therefore, the distance is given by $|\frac{19}{49}(3, 2, 6)| = \frac{19}{7}$.

4. How far is $b = (1, 2, 2)$ from the plane $Ax + By + Cz = D$?

solution

The normal direction of the plane is given by $n = (A, B, C)$. Choose arbitrary (x_0, y_0, z_0) in the plane. Then $b - (x_0, y_0, z_0)$ is a vector from the plane to the point. We can project $b - (x_0, y_0, z_0)$ onto the normal direction to get

$$\begin{aligned}\frac{(b-(x_0,y_0,z_0))\cdot n}{|n^2|}n &= \frac{A+2B+2C-(Ax_0+By_0+Cz_0))}{A^2+B^2+C^2}(A,B,C)\\ &= \frac{A+2B+2C-D}{A^2+B^2+C^2}(A,B,C)\end{aligned}$$

Therefore, the distance is given by $|\frac{A+2B+2C-D}{A^2+B^2+C^2}(A,B,C)| = \frac{|A+2B+2C-D|}{\sqrt{A^2+B^2+C^2}}$.

5. Suppose that $A'Ax = 0$. Show that $Ax = 0$.
Hint: One approach is in Problem 4.1.9: We are told that Ax is in the nullspace of A'. Which space does Ax also lie in?

solution

Ax lies in the column space of A by the definition of column space. Since $A'Ax = 0$, Ax lies in the null space of A', i.e. It lies in the left null space of A. Since the left null space of A is perpendicular to the column space of A, we get Ax is perpendicular to itself. Therefore, $Ax = 0$.

6. Find the 3 by 3 matrix P that projects every vector b in $\mathbb{R}^3$ onto the plane $x+2y+2z = 0$. Pb is the closest point in the plane to b.

solution

The normal direction of the plane is $n = (1, 2, 2)$. For any vector v, the projection onto the normal direction is given by $\frac{n\cdot v}{|n|^2}n$. Therefore, the matrix of projection onto the normal direction is given by

$$M = \frac{1}{1^2+2^2+2^2}\begin{bmatrix}1\\2\\2\end{bmatrix}\begin{bmatrix}1 & 2 & 2\end{bmatrix} = \frac{1}{9}\begin{bmatrix}1 & 2 & 2\\ 2 & 4 & 4\\ 2 & 4 & 4\end{bmatrix}.$$

Therefore, projection onto the plane is given by

$$I_3 - M = \begin{bmatrix}\frac{8}{9} & -\frac{2}{9} & -\frac{2}{9}\\ -\frac{2}{9} & \frac{5}{9} & -\frac{4}{9}\\ -\frac{2}{9} & -\frac{4}{9} & \frac{5}{9}\end{bmatrix}.$$

7. Problem 13 of Section 4.2. Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $b = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

solution

The matrix P is a 4 by 4 square matrix. Since all the column of A are independent. We can use the equation

$$P = A(A'A)^{-1}A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The projection of b on the column space of A is given by $Pb = (1, 2, 3, 0)$.

8. Problem 31 of Section 4.2. In $\mathbb{R}^m$, suppose I give you b and p, and n linearly independent vectors $a_1, \dots, a_n$. How would you test to see if p is the projection of b onto the subspace spanned by the a's? A be the m by n matrix of column vectors $a_1, \dots, a_n$. For p to be the projection of b, we need to check 2 conditions: (1) The vector p lies in the column space of A. (2) The vector $b - p$ lies in the left null space of A. Since the columns of A are linearly independent, we can also check if $p = A(A'A)^{-1}A'b$.

9. Construct matrices with the following properties. Write None if no such matrix can be constructed and explain why. (Explanation should be in the form of a matrix type equation.)

   (a) Column space contains $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, nulspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

   (b) row space contains $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, nulspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(c) $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has solution and $A^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(d) Every row is orthogonal to every column. (A is not the zero matrix.)

(e) Columns add up to a column of zeros. Rows add to a row of 1s.

solution

(a) Let $a_1 = (1, 2, -3)^T$, $a_2 = (2, -3, 5)^T$. Since they are independent, we have $\dim C(A) \geq 2$. Since nullspace contains $(1, 1, 1)^T$, $\dim N(A) \geq 1$. Together with $\dim C(A) + \dim N(A) = 3$. We have $C(A) = \text{span}\{a_1, a_2\}$ and $N(A) = \text{span}(1, 1, 1)^T$. To construct such a matrix, we can let the first column to be a_1 and the second column to be a_2. Since $(1, 1, 1)^T$ is in the null space, we must have the sum of three columns is a zero column. Therefore, a matrix satisfying the property is

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}.$$

(b) For any matrix, row space is perpendicular to the null space. However, $(2, -3, 5)^T$ is not perpendicular to $(1, 1, 1)^T$. Therefore, there cannot be such matrix. Answer: **None**.

(c) For any matrix, column space is perpendicular to the left null space. However, we know $(1, 1, 1)^T$ is in the column space, $(1, 0, 0)^T$ is in the left null space and the vectors are not perpendicular. Therefore, there cannot be such matrix. Answer: **None**.

(d) From the condition that every row is orthogonal to every column. We know A must be a square matrix satisfying $A^2 = 0$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is a solution.

(e) Let A be an m by n matrix and a_{ij} be its i, j-entry. Since the column vectors add up to a column of zeros, we have for all i, $\sum_{j=1}^{n} a_{ij} = 0$. Therefore, we have $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = 0$. On the other hand, since the rows add to a row of 1s, we have for all j, $\sum_{i=1}^{m} a_{ij} = 1$. Therefore, we have $\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} = n$. Since $n \neq 0$, the matrix cannot exist. Answer: **None**.

10. (a) Suppose you are given nonzero column vectors $\mathbf{r}, \mathbf{n}, \mathbf{c}, \mathbf{l}$ in $\mathbb{R}^2$. Explain how to determine if these vectors can form bases for the 4 fundamental subspaces, row space, nullspace, column space and left nullspace respectively, and outputs a matrix with those 4 fundamental subspaces if possible. Hint: Think about orthogonality of the four subspaces. Express matrix in terms of the vectors above.

(b) Suppose that I give you 4 matrices whose columns are all vectors in $\mathbb{R}^{10}$. $R = \begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_i\end{bmatrix}, N = \begin{bmatrix}\mathbf{n}_1 & \mathbf{n}_2 & \cdots & \mathbf{n}_j\end{bmatrix}, C = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_i\end{bmatrix}, L = \begin{bmatrix}\mathbf{l}_1 & \mathbf{l}_2 & \cdots & \mathbf{l}_n\end{bmatrix}$. The columns of R form a basis for the row space, the columns of N form a basis for the nullspace, the columns of C form a basis for the column space, and the columns of L form a basis for the left nullspace. Explain how to determine if these vectors form the basis for the four fundamental subspaces, ad output a matrix that has those four subspaces.

(c) Suppose that I give you 4 matrices: $R = \begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_i\end{bmatrix}, N = \begin{bmatrix}\mathbf{n}_1 & \mathbf{n}_2 & \cdots & \mathbf{n}_j\end{bmatrix}$, $C = \begin{bmatrix}\mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_m\end{bmatrix}$, $L = \begin{bmatrix}\mathbf{l}_1 & \mathbf{l}_2 & \cdots & \mathbf{l}_n\end{bmatrix}$. The dimensions are not specified. Do the same things as above. Note the dimension of the output matrix.

solution

(a) By the orthogonality of the fundamental subspaces, we must have $\mathbf{r}\cdot\mathbf{n} = \mathbf{r^T n} = 0$, $\mathbf{c}\cdot\mathbf{l} = \mathbf{c^T l} = 0$. If the 4 column vectors satisfies the above condition, $A = \mathbf{cr^T}$ is a matrix satisfies the condition.

(b) Use the same idea above. By the orthogonality of the fundamental subspaces, we must have $R^T N = 0$, $C^T L = 0$. We also need the dimension to match, i.e. $i + j = i + n = 10$. Under the condition above, $A = CR^T$ is a matrix satisfies the condition.

(c) We need to deal with which space does each of the subspace lies in. Therefore, we need the height of R and N be the same. We also need the height of C and L be the same. After checking the height, by the orthogonality of the fundamental subspaces, we must have $R^T N = 0$, $C^T L = 0$. We also need the dimension to match, $i = m$(row rank equals column rank). Height$(N) = j + m$. Height$(L) = i + n$. Under the condition above, $A = CR^T$ is a matrix satisfies the condition.