Problem Set 8, 18.06

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This problem set is due Thursday, April 24, 2014 by 4pm in E17-131. This homework has 9 questions worth 90 points in total. Please WRITE NEATLY. You may discuss with others (and your TA), but you must turn in your own writing.

Problem 1 (6.4 ♯5). Find an orthogonal matrix Q that diagonalizes the symmetric matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}.$$

Problem 2 (6.4 ♯10). Here is a quick "quick" proof that the eigenvalues of all real matrices are real:

False proof $Ax = \lambda x$ gives $x^T Ax = \lambda x^T x$ so $\lambda = \dfrac{x^T Ax}{x^T x}$ is real.

Find the flaw in this reasoning—a hidden assumption that is not justified. You could test those steps on the 90° rotation matrix $[0 \;\; -1; 1 \;\; 0]$ with $\lambda = i$ and $x = (i, 1)$.

Problem 3 (6.4 ♯23). Which of these classes of matrixes do A and B belong to: Invertible, orthogonal, projection,permutation, diagonalizable, Markov?

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Which of these factorizations are possible for A and B: LU, QR, $S\Lambda S^{-1}$, $Q\Lambda Q^{-1}$?

Problem 4 (6.5 ♯7). Test to see if $R^T R$ is positive definite in each case:

$$R = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Problem 5 (6.5 ♯12). For what numbers c and d are A and B positive definite? Test the 3 determinants:

$$A = \begin{pmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{pmatrix}.$$

Problem 6 (6.5 ♯14). If A is positive definite then A^{-1} is positive definite.

Best proof: The eigenvalues of A^{-1} are positive because _____.

Second proof (only for 2 by 2): The entries of $A^{-1} = \frac{1}{ac-b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$ pass the determinant tests _____.

Problem 7 (6.5 ♯19). Suppose that all the eigenvalues λ of a diagonalizable matrix A satisfy that $\lambda > 0$. Show that then $x^T Ax > 0$ for every nonzero vector x. Note that x is not necessarily an eigenvector of A, so write x as a linear combination of these eigenvectors and explain why all the "cross terms" are $x_i^T x_j = 0$. Then, argue that $x^T Ax$ is

$$(c_1 x + \cdots + c_n x_n)(c_1\lambda_1 x_1 + \cdots + c_n\lambda_n x_n) = c_1^2\lambda_1 x_1^T x_1 + \cdots + c_n^2\lambda_n x_n^T x_n > 0.$$

Problem 8 (6.5 ♯28). Without multiplying

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

find:

a) the determinant of A,

b) the eigenvalues of A,

c) the eigenvectors of A,

d) the reason why A is symmetric positive definite.

Problem 9 (Matlab, 2pt) Explain why $B_{13}^{-1} = B_{31}^{-1} = 0$ in part 1 and submit it with the written parts.