18.06 Exam I Professor Strang March 7, 2014

Solutions

Question 3.(c) is a 1 point question. All other questions are worth 11 points each.

1. Suppose the blocks in A are 3 by 3 (so A is 6 by 6), and $F = ones(3)$ is the all-ones matrix:

$$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(a) Find a basis for the nullspace $N(A)$.

(b) Find a basis for the left nullspace $N(A^T)$.

(c) Exactly which matrices have dimension of nullspace of A equal to dimension of nullspace of A^T ?

Solution.

(a) Matrix A is already in its $rref$. There are three pivots. Therefore, $r = \dim C(A) = \dim R(A) = 3$. Hence, $\dim N(A) = 6 - r = 3$. The special basis is $[-1, -1, -1, 1, 0, 0]'$, $[-1, -1, -1, 0, 1, 0]'$, and $[-1, -1, -1, 0, 0, 1]'$.

(b) The dimension of the left nullspace is: $\dim N(A^T) = 6 - r = 3$, with a basis: $[0, 0, 0, 1, 0, 0]'$, $[0, 0, 0, 0, 1, 0]'$, and $[0, 0, 0, 0, 0, 1]'$.

(c) The dimension of the nullspace of A is $n - r$, the dimension of the left nullspace is $m - r$. They are equal when $n - r = m - r$. That is, when $m = n$. The dimensions are equal when the matrix is square.

2. **(a)** What value of q gives A a different rank compared to all other values of q?

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 3 & 4 \\ 4 & 3 & 9 & q \end{bmatrix}$$

(b) With that special value of q, what are the conditions on b_1, b_2, b_3 for $Ax = b$ to have a solution?

(c) If those conditions are satisfied by b_1, b_2, b_3, what are all the solutions x (the complete solution to $Ax = b$ with that special value of q) ?

Solution.

(a) Start the elimination: replace row 2 (r_2) with $r_1 - 2r_2$, then replace r_3 with $r_3 - 4r_1$ to get:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -5 & -3 & 2 \\ 0 & -5 & -3 & q-4 \end{bmatrix}.$$

Continue by replacing the third row with the difference of the third minus the second row to get the triangular matrix U:

$$U = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -5 & -3 & 2 \\ 0 & 0 & 0 & q-6 \end{bmatrix}.$$

When $q = 6$, U has two pivots, and the rank of A is 2. Otherwise, the rank of A is 3.

(b) Repeat the elimination steps on $b = (b_1, b_2, b_3)$ to get $(b_1, b_2 - 2b_1, b_3 - b_2 - 2b_1)$. The condition is for the last coordinate to be zero: $b_3 - b_2 - 2b_1 = 0$.

(c) There are two pivots and two free variables: x_3 and x_4. The nullspace is 2-dimensional with special solutions: $[-9/5, -3/5, 1, 0]'$ and $[-9/5, 2/5, 0, 1]'$. We can get a particular solution when we assign free variables to be zero, and solve for the pivot variables:

$$\left[\begin{array}{rcrcr} x_1 & + & 2x_2 & = & b_1 \\ & & -5x_2 & = & b_2 - 2b_1 \end{array}\right].$$

The result is: $[b_1/5{+}2b_2/5, -b_2/5{+}2b_1/5, 0, 0]'$. The complete solution is: $[b_1/5{+}2b_2/5, -b_2/5{+}2b_1/5, 0, 0]'{+}c[-9/5, -3/5, 1, 0]'{+}d[-9/5, 2/5, 0, 1]'$.

3. Suppose the nullspace of A (5 by 4 matrix) is spanned by v and w, which are special solutions to $Ax = 0$:

$$v = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

(a) What is the row reduced echelon form $R = rref(A)$? We don't have to know A.

(b) Which vectors in R^4 can be rows of A? How many of the 5 rows of A will be independent?

(c) ***One point question:*** What is the dimension of the matrix space containing all 5 by 4 matrices A that have those vectors v and w in their nullspace?

(d) If C is any 4 by 7 matrix of rank $r = 4$, find the column space of C. Explain clearly why $Cx = b$ always has infinitely many solutions.

Solution.

(a) It is clear that x_2 and x_4 are free variables. We can reconstruct a part of R right away:

$$R = \begin{bmatrix} 1 & a & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find a, b, and c we can use the fact that $r_1 \cdot v = r_1 \cdot w = r_2 \cdot v = r_2 \cdot w = 0$. We get $a = -4$, $b = -1$, $c = -2$. Thus,

$$R = \begin{bmatrix} 1 & -4 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) The dimension of the nullspace is 2 and is equal to $n - r$. So $r = 4 - 2 = 2$, Therefore, the dimension of the row space is 2. Thus, exactly two rows in A are independent. The row space of A is the same as the row space of R. So any row in A must be a linear combination of the first two rows of R: $c[1, -4, 0, -1] + d[0, 0, 1, -2]$.

(c) Each row can be any vector in a 2-dimensional space orthogonal to v and w. There are five rows. So the dimension of the space of all such matrices is 10.

(d) The rank of the matrix is equal to the dimension of the column space. Thus the dimension of the column space is 4. Therefore, the column space spans all of R^4. A basis of the column space is $[1, 0, 0, 0]'$, $[0, 1, 0, 0]'$, $[0, 0, 1, 0]'$ and $[0, 0, 0, 1]'$. Hence, for any vector b there exists a solution. In addition, the dimension of the nullspace is 3. Therefore, there are infinitely many solutions.