18.06 Exam III Professor Strang May 7, 2014

Your PRINTED Name is: ______________________________

Please circle your section:

R01	T	10	36-144	Qiang Guang
R02	T	10	35-310	Adrian Vladu
R03	T	11	36-144	Qiang Guang
R04	T	11	4-149	Goncalo Tabuada
R05	T	11	E17-136	Oren Mangoubi
R06	T	12	36-144	Benjamin Iriarte Giraldo
R07	T	12	4-149	Goncalo Tabuada
R08	T	12	36-112	Adrian Vladu
R09	T	1	36-144	Jui-En (Ryan) Chang
R10	T	1	36-153	Benjamin Iriarte Giraldo
R11	T	1	36-155	Tanya Khovanova
R12	T	2	36-144	Jui-En (Ryan) Chang
R13	T	2	36-155	Tanya Khovanova
R14	T	3	36-144	Xuwen Zhu
ESG	T	3		Gabrielle Stoy

Grading 1:

2:

3:

4:

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1. (28 points) This question is about the differential equation

$$\frac{dy}{dt} = Ay = \begin{bmatrix} 5 & 2 \\ 8 & 5 \end{bmatrix} y \quad \text{with } y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) Find an eigenvalue matrix Λ and an eigenvector matrix S so that $A = S\Lambda S^{-1}$. Compute the matrix exponential e^{tA} by using $e^{t\Lambda}$.

(b) Find $y(t)$ as a **combination of the eigenvectors** of A that has the correct value $y(0)$ at $t = 0$.

Solutions:

(a) $\det(A - \lambda I) = 0 \Leftrightarrow \lambda^2 - 10\lambda + 9 = 0$. Eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 9$. The eigenvector associated to λ_1 is $v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and the eigenvector associated to λ_2 is $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The matrix $S = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$ and $S^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$. Finally, $e^{tA} = Se^{t\Lambda}S^{-1} = \begin{pmatrix} \frac{1}{2}e^t + \frac{1}{2}e^{9t} & -\frac{1}{4}e^t + \frac{1}{4}e^{9t} \\ -e^t + e^{9t} & -\frac{1}{2}e^t + \frac{1}{2}e^{9t} \end{pmatrix}$.

(b) $y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ -2 \end{pmatrix} + b\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This implies that $a = \frac{1}{2}$ and $b = \frac{1}{2}$. Hence, $y(t) = ae^{\lambda_1 t}v_1 + be^{\lambda_2 t}v_2 = \frac{1}{2}e^t\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{1}{2}e^{9t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

2. (a) (24 points) Suppose a symmetric n by n matrix S has eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ and orthonormal eigenvectors $q_1, \ldots, q_n$.

If $x = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n$ show that $x^T x = c_1^2 + \cdots + c_n^2$ and $x^T S x = \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2$.

(b) What is the largest possible value of $R(x) = \frac{x^T S x}{x^T x}$ for nonzero x ?

Describe a vector x that gives this maximum value for this ratio $R(x)$?

Solutions:

(a) Since the eigenvectors are orthonormal, one has $x^T x = (c_1 q_1 + \cdots + c_n q_n)^T (c_1 q_1 + \cdots + c_n q_n) = c_1^2 q_1^T q_1 + \cdots + c_n^2 q_n^T q_n = c_1^2 + \cdots + c_n^2$. On the other hand, $x^T S x = (c_1 q_1 + \cdots + c_n q_n)^T S (c_1 q_1 + \cdots + c_n q_n) = (c_1 q_1 + \cdots + c_n q_n)^T (\lambda_1 c_1 q_1 + \cdots + \lambda_n c_n q_n) = \lambda_1 c_1^2 q_1^T q_1 + \cdots + \lambda_n c_n^2 q_n^T q_n = \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2$.

(b) Using (a), $R(X) = \frac{\lambda_1 c_1^2 + \cdots + \lambda_n c_n^2}{c^1 + \cdots + c_n}$. Since $\lambda_1 > \lambda_2 > \cdots > \lambda_n$, $R(X)$ is maximal when $c_2 = \cdots = c_n = 0$ and $c_1 \neq 0$. In this case the largest value of $R(x)$ is λ_1 and the associated vector x is any non-zero multiple of q_1.

3. (24 points)

(a) Show that the matrix $S = A^T A$ is positive semidefinite, for any matrix A. Which test will you use and how will you show it is passed?

(b) If A is 3 by 4, show that $A^T A$ is **not** positive definite.

Solutions:

(a) Energy test. For every vector x we have $x^T S x = x^T A^T A x = (Ax)^T (Ax) = ||Ax||^2 \geq 0$. Hence, S is positive semidefinite.

(b) Since A is 3×4, one has $\dim(C(A)) \leq 3$ and $\dim(N(A)) \geq 1$. Hence, there exists a non-zero vector v such that $Av = 0$. As a consequence, $A^T A$ is **not** positive definite.

4. (24 points)

(a) Show that none of the singular values of A are larger than 3.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Why does $B = AQ$ have the same singular values as A? (Q is an orthogonal matrix.)

Solutions:

(a) $A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$. Hence, $\mathrm{tr}(A^T A) = 6$. However, $A^T A$ is positive semidefinite, therefore all the eigenvalues are nonnegative. This implies that $0 \leq \lambda_i \leq 6$ and hence that $\sigma_i \leq \sqrt{6} \leq 3$.

(b) Since $B^T B = Q^T A^T A Q$, the matrixes $B^T B$ and $A^T A$ are similar. This implies that they have the same eingenvalues and therefore that B and A have the same singular values $\sigma_i = \sqrt{\lambda_i}$.

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