18.06 Problem Set 6 - SOLUTIONS

Problem 1.
Let A be an n by n square matrix and let v be the n by 1 vector of all ones. Then if the entries in every row of A add to zero, $Av = 0$, so A is not invertible and $\det A = 0$. Similarly, if the entries in every row of A add to one, $(A - I)v = 0$, so $A - I$ is not invertible and $\det(A - I) = 0$.

Problem 2.

(a) $C_1 = 0, C_2 = -1, C_3 = 0, C_4 = 1$.
(b) By taking the cofactor expansion along the first row, we obtain $C_n = -C_{n-2}$, so $C_{2n} = (-1)^n$ and $C_{2n+1} = 0$. Thus $C_{10} = -1$.

Problem 3.

For A: Taking the cofactor expansion along the second row, we compute $\det A = b$.

For B: Recall that subtracting a multiple of a row from another row does not change the determinant. Perform the following three row operations in order: subtract a times the third row from the last row, subtract a times the second row from the third row, and subtract a times the first row from the second row. This yields the matrix

$$\begin{bmatrix} 1 & a & a^2 & a^3 \\ 0 & 1-a^2 & a-a^3 & a^2-a^4 \\ 0 & 0 & 1-a^2 & a-a^3 \\ 0 & 0 & 0 & 1-a^2 \end{bmatrix}$$

which has determinant $(1 - a^2)^3 = 1 - 3a^2 + 3a^4 - a^6 = \det B$.

Problem 4.

(a) The subspace spanned by the last three rows is at most two-dimensional.
(b) Let x be any term in the big formula for $\det A$, so up to sign

$$x = a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}a_{4\sigma(4)}a_{5\sigma(5)}$$

for some permutation σ of $\{1, ..., 5\}$. (The sign equals $\det P$ where P is the permutation matrix with $P_{i\sigma(i)} = 1$ and zero entries elsewhere). Then since $\sigma(3), \sigma(4), \sigma(5)$ are distinct, at least one of $a_{3\sigma(3)}, a_{4\sigma(4)}, a_{5\sigma(5)}$ must be zero, so $x = 0$.

Problem 5.

Let A be a m by m square matrix such that $|\det(A)| > 1$. Suppose for a contradiction that there exists a constant C such that $|(A^n)_{ij}| \leq C$ for all $1 \leq i, j \leq m$. Then

$$|\det(A^n)| = \left|\sum_\sigma \det(\sigma)(A^n)_{1\sigma(1)} \ldots (A^n)_{m\sigma(m)}\right| \leq \sum_\sigma C^m = m!\, C^m$$

where the sum is over all permutations σ of $\{1, ..., m\}$ and $\det(\sigma)$ is the determinant of the associated permutation matrix. But $|\det(A^n)| = |\det(A)|^n \to \infty$ as $n \to \infty$, a contradiction. To put it in words, the determinant of a matrix is a polynomial in the entries of a matrix, so if all those entries are bounded then the determinant is bounded. However, the determinant of A^n goes (in absolute value) to infinity as $n \to \infty$, so we cannot have a bound on all the entries.

For the second part of the problem, $A = \begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}$ is an example of a matrix with $\det(A) = 1/2$ but $(A^n)_{11} = 2^n \to \infty$ as $n \to \infty$.