18.06.02: ‘Vectors’

Lecturer: Barwick

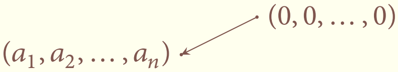
Wednesday 05 February 2016

Lines vs. vectors

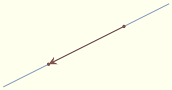
A vector is a list of real numbers

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

We draw this vector as an arrow pointing from the point $(0, 0, \ldots, 0)$ to the point $(a_1, a_2, \ldots, a_n)$:



By extending this vector, we obtain a line:



This is the unique line between the origin and $(a_1, a_2, \ldots, a_n)$.

Question. How can we write a formula for a parametrization of this line?

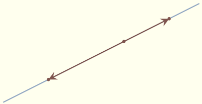
$\lambda(t) = (ta_1, ta_2, \ldots, ta_n)$.

Note that there's one exception to this: the *zero vector*

$$\vec{0} \coloneqq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

doesn't give us much of a line. This is the *only* special vector. We'll return to this.

Two vectors may determine the same line:



Question. When does this happen?

If there's a real number r such that

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} rb_1 \\ rb_2 \\ \vdots \\ rb_n \end{pmatrix},$$

then we say that $\vec{a}$ is a *scalar multiple* of $\vec{b}$, and we just write $\vec{a} = r\vec{b}$. In this case, as long as $\vec{a}$ and $\vec{b}$ are nonzero, they determine the same line.

So lines through the origin are the "same thing" as nonzero vectors, up to scaling.

$\mathbf{R}^1$ and $\mathbf{R}^2$

We write $\mathbf{R}^n$ for the collection of all vectors

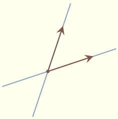
$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Thus $\mathbf{R}^1$ is simply the set of real numbers, but we regard those real numbers as arrows from the origin to the number as it sits on the number line:

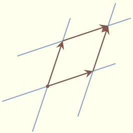
Note that if $a \in \mathbf{R}^1$ is a nonzero real number, then any real number is a scalar mutiple of a. In other words, $\mathbf{R}^1$ is 1-dimensional.

How about $\mathbf{R}^2$? This is the set of vectors $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. This is the set of vectors that point from the origin to points on the plane.

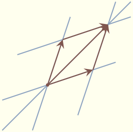
A nonzero vector $\vec{a} \in \mathbf{R}^2$ specifies a line through the origin. Two vectors $\vec{a}, \vec{b} \in \mathbf{R}^2$ specify two *distinct* lines through the origin if and only if there exists no real number r such that $\vec{a} = r\vec{b}$:



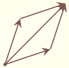
Now remember that we translated each line along the other to end up with a parallelogram. Let’s do that here in such a way that the vectors meet head to tail:



We thus get a new vector that points at the new vertex we created.



Now let’s clear out the lines spanned by these vectors, and gaze lovingly at the vectors themselves:



What we’ve drawn there is a way of taking two vectors $\vec{a}$ and $\vec{b}$, translating $\vec{a}$ along $\vec{b}$ and translating $\vec{b}$ along $\vec{a}$ to make a parallelogram. The diagonal vector of that parallelogram is

$$\vec{a} + \vec{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}.$$

This is *vector addition.*

The fact that you can get that diagonal by translating $\vec{a}$ along $\vec{b}$ or by translating $\vec{b}$ along $\vec{a}$ is way of visualizing the commutativity of vector summation. It turns out that *all* the algebraic properties of vectors have pictures to go along with them.

Question. What does associativity look like? adding with $\vec{0}$? the formation of negatives? the distribution of scalar multiplication over vector addition?

$\mathbf{R}^3$

Of course $\mathbf{R}^3$ is the set of all vectors $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$. Consider two vectors $\vec{a}, \vec{b} \in \mathbf{R}^3$ such that there exists no real number r such that $\vec{a} = r\vec{b}$. These two vectors define a plane: the vectors that can be writen as a *linear combination* $s\vec{a} + t\vec{b}$; we call this plane the *span* of $\vec{a}$ and $\vec{b}$. So a vector $\vec{c}$ does not lie in the *span* of $\vec{a}$ and $\vec{b}$ if and only if it can't be written as a linear combination of of $\vec{a}$ and $\vec{b}$.

Once we've found such a $\vec{c}$, I claim that any vector $\vec{v} \in \mathbf{R}^3$ can be written as a linear combination of $\vec{a}$, $\vec{b}$, and $\vec{c}$. Geometrically, this means any point (v_1, v_2, v_3) lies in the span of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

Algebraically, this means that for any $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, there is a solution (r, s, t) to the equation $\vec{v} = r\vec{a} + s\vec{b} + t\vec{c}$. But that equation is secretly the system of linear equations

$$\begin{aligned} v_1 &= ra_1 + sb_1 + tc_1; \\ v_2 &= ra_2 + sb_2 + tc_2; \\ v_3 &= ra_3 + sb_3 + tc_3. \end{aligned}$$

What we’re saying is that this set of equations has a solution.

But there's more: the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are *linearly indepenedent* – that is, the lines they span are independent.

What this means algebraically is that none of them are zero, and there's no way to write

$$\vec{c} = s\vec{a} + t\vec{b}$$

or

$$\vec{b} = p\vec{a} + q\vec{c}$$

or

$$\vec{a} = m\vec{b} + n\vec{c}.$$

We can express this more efficiently: what it means for $\vec{a}$, $\vec{b}$, and $\vec{c}$ to be linearly indepenedent is that if

$$r\vec{a} + s\vec{b} + t\vec{c} = \vec{0},$$

then $r = s = t = 0$.

What we will learn is that for any $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, there is a *unique* solution (r, s, t) to the equation $\vec{v} = r\vec{a} + s\vec{b} + t\vec{c}$.

Let's do an example. We'll find three independent vectors. We have to start with the nonzero vector $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. For our next vector, we just need to select a vector that is not a multiple of $\vec{a}$. Here's one: $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

Now for the tricky bit. We want a third vector, $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, that doesn't lie in the plane spanned by $\vec{a}$ and $\vec{b}$. That's more than just making sure that $\vec{c}$ is not a scalar multiple of $\vec{a}$ or $\vec{b}$.

OK, does $\begin{pmatrix} -5/2 \\ 15 \\ 15/2 \end{pmatrix}$ lie in the plane spanned by $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$??

It looks like it does!

$$\frac{15}{2}\begin{pmatrix}1\\2\\1\end{pmatrix} + (-5)\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-2/5\\15\\15/2\end{pmatrix}.$$

So how do we make sure that we get a vector that doesn't live on that plane?

Here's one that will work: $\begin{pmatrix}0\\0\\2\end{pmatrix}$. What makes me so sure that will work?

See, a linear combination of $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ that gives $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ would have to have a nonzero coefficient on $\vec{a}$, and that will give a nonzero second coordinate.

So we have our example:

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \vec{c} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

are three linearly independent vectors in $\mathbf{R}^3$.

Now my claim is that any vector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ can be written as a *unique* linear combination of $\vec{a}$, $\vec{b}$, and $\vec{c}$. If $\vec{v} = \begin{pmatrix} 2 \\ -6 \\ 14 \end{pmatrix}$, for example, we're trying to solve this system of linear equations:

$$\begin{aligned} 2 &= r + 2s + 0t; \\ -6 &= 2r + 0s + 0t; \\ 14 &= r + 0s + 2t. \end{aligned}$$

So we straight away get $r = -3$, $s = 5/2$, and $t = 17/2$.

Here is the general algebraic picture:

Definition. If $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$ are vectors of $\mathbf{R}^n$, then the *span* of $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$ is the set of all linear combinations

$$r_1\vec{a}_1 + r_2\vec{a}_2 + \ldots + r_k\vec{a}_k.$$

So $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ lies in the span of $\vec{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \ldots, \vec{a}_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix}$
if and only if the system of linear equations

$$\begin{aligned} v_1 &= r_1 a_{11} + r_2 a_{12} + \cdots + r_k a_{1k}; \\ v_2 &= r_1 a_{21} + r_2 a_{22} + \cdots + r_k a_{2k}; \\ &\vdots \\ v_n &= r_1 a_{n1} + r_2 a_{n2} + \cdots + r_k a_{nk}. \end{aligned}$$

has at least one solution.

Geometrically, that means that a vector $\vec{v}$ lies in the span of $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_k$ when you can build a multidimensional parallelopiped out of $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_k$ to get $\vec{v}$ pointing from the origin across the diagonal.

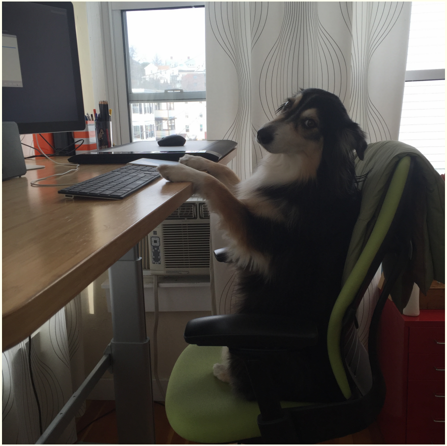


Figure 1: Doesn’t know how to make an image of multidimensional paralellopipeds.

Definition. We say vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$ of $\mathbf{R}^n$ are *linearly independent* if we're in the following situation: any *vanishing* linear combination

$$r_1\vec{a}_1 + r_2\vec{a}_2 + \ldots + r_k\vec{a}_k = \vec{0}$$

must be a *trivial* linear combination – that is, we must have

$$r_1 = r_2 = \cdots = r_k = 0.$$

So the vectors $\vec{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \ldots, \vec{a}_k = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix}$ are linearly independent if and only if any system of linear equations

$$\begin{aligned} v_1 &= r_1 a_{11} + r_2 a_{12} + \cdots + r_k a_{1k}; \\ v_2 &= r_1 a_{21} + r_2 a_{22} + \cdots + r_k a_{2k}; \\ &\vdots \\ v_n &= r_1 a_{n1} + r_2 a_{n2} + \cdots + r_k a_{nk}. \end{aligned}$$

has at most one solution.

Geometrically, that means that none of the vectors $\vec{a}_i$ lie in the span of the remaining vectors

$$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{i-1}, \vec{a}_{i+1}, \ldots, \vec{a}_k.$$

Figure 2: Cool. Can we go play in the snow now?

We'd like a more efficient way of checking whether a vector lies in a span of a certain collection of vectors and whether that collection of vectors is linearly independent.

To do this, we'll want to introduce two fundamental manipulations of vectors: forming the dot product of two vectors and combining a list of vectors into a matrix. The dot product extracts a bunch of helpful geometry, and matrices give us an efficient way to organize and conceptualize systems of linear equations.

For next week, please read §§1.2–2.1 of Strang.

The first problem set will be posted on Monday.