18.06.12: 'Kernels and images'

Lecturer: Barwick

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Suppose we can apply some row operations:

$$(A \mid B) \dashrightarrow (C \mid D).$$

Here, *A* and *C* are $m \times n$ matrices, and *B* and *D* are $m \times p$ matrices. What this really means is that there's an invertible $m \times m$ matrix *M* such that MA = C and MB = D. (And it turns out that any *M* can be built this way!)



So you can use row operations to whittle your favorite matrix down, and then solve.

First, let's apply row operations to

$$(A|0) = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 & 0\\ 1 & -2 & 2 & 3 & -1 & 0\\ 2 & -4 & 5 & 8 & -4 & 0 \end{pmatrix}?$$



When we get *A* into *reduced row echelon form* (rref, as the cool kids say) we get

Why is this good? First, we haven't changed the kernel. A vector $\vec{x} \in \mathbb{R}^5$ is in ker(*A*) if and only if \vec{x} is in ker(*C*) = ker(*MA*). (Why?)



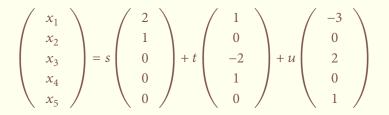
Second, we can use the rref above gives us this system of linear equations

$$x_1 = 2x_2 + x_4 - 3x_5$$

$$x_3 = -2x_4 + 2x_5$$

So x_1 and x_3 can each be written in terms of x_2, x_4, x_5 , and there's no dependence among x_2, x_4, x_5 . So pick variables s, t, u, and let $x_2 = s, x_4 = t$, and $x_5 = u$.





There's your basis!



Let's find the kernel via row reduction

$$A = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$



When we get (A|0) into rref, we obtain

$$\left(\begin{array}{ccccccccccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$



So if
$$x_2 = s$$
 and $x_5 = t$, then

$$\begin{pmatrix} x_{1} \\ x_{2} \\ x_{3} \\ x_{4} \\ x_{5} \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

There's your basis!



Column operations work exactly dual to row operations. (Just think of transposing, doing row operations, and transposing back!) So suppose we can apply some column operations:

$$\left(\frac{A}{B}\right) \dashrightarrow \left(\frac{C}{D}\right).$$

Here, *A* and *C* are $m \times n$ matrices, and *B* and *D* are $p \times n$ matrices. What this really means is that there's an invertible $n \times n$ matrix *N* such that AN = C and BN = D.

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Why is that a good idea? Well, we're looking for vectors such that $A\vec{x} = \vec{0}$. So if we take

where *I* is the $n \times n$ identity matrix, then we can start using column operations to get it to some

So AD = C. So if *C* has a column of zeroes, then the corresponding column of *D* will be a vector in the kernel. Furthermore, if you get *C* into column echelon form, then the nonzero column vectors of *D* lying under the zero columns of *C* form a basis of ker(*A*). (Properly speaking, to prove this, you need the







Rank-Nullity theorem, which we'll come to soon.)



Let's do this one:

Let us get the top of
$$\left(\frac{A}{I}\right)$$
 into column echelon form



We obtain

The last three columns of *D* are our basis.



Dual to finding a basis of the kernel, we can find a basis of the *image* of a matrix, im(A). The image of A is the span of the column vectors of A.

If *A* is an $m \times n$ matrix, then *A* eats a vector of \mathbb{R}^n , and it poops a vector of \mathbb{R}^m . The kernel of *A* is thus a subspace of \mathbb{R}^n , and the image of *A* is a subspace of \mathbb{R}^m .

The way we *compute* a basis of the image is not wildly different from the way in which we compute a basis of the kernel, but the operations are dual, and that can get confusing. To clear up our confusion, we'll need some theorems!!



Theorem (Rank-Nullity Theorem). If A is an $m \times n$ matrix, then

 $\dim(\ker(A)) + \dim(\operatorname{im}(A)) = n.$

We are going to spend some quality time with this result.