18.06.13: 'Kernels and images – the sequel'

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Column operations work exactly dual to row operations. (Just think of transposing, doing row operations, and transposing back!) So suppose we can apply some column operations:

$$\left(\frac{A}{B}\right) \rightsquigarrow \left(\frac{C}{D}\right).$$

Here, A and C are $m \times n$ matrices, and B and D are $p \times n$ matrices. What this really means is that there's an invertible $n \times n$ matrix N such that $AN = C$ and $BN = D$.

Well, we’re looking for vectors such that $A\vec{x} = \vec{0}$. So if we take

$$\left(\frac{A}{I}\right),$$

then we can start using column operations to get it to some

$$\left(\frac{C}{D}\right)$$

So $AD = C$.

So when we look at

$$\left(\frac{C}{D}\right),$$

if C has a column of zeroes, then the corresponding column of D will be a vector in the kernel. Furthermore, if you get C into column echelon form, then the nonzero column vectors of D lying under the zero columns of C form a basis of $\ker(A)$. (Properly speaking, to prove this, you need the Rank-Nullity theorem, which we'll come to soon.)

Let's do this one:

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 & 2 & -8 \\ 0 & 1 & 5 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & 7 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let us get the top of $\left(\frac{A}{I}\right)$ into column echelon form.

We obtain

$$\left(\frac{C}{D}\right) = \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 3 & -2 & 8 \\ 0 & 1 & 0 & -5 & 1 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -7 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

The last three columns of D are our basis.

If

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix},$$

we already found a basis of $\ker(A)$:

$$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

To use our column method, we don't have to get the top in reduced column echelon form

$$\left(\begin{array}{ccccc} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

We obtain:

$$\left(\begin{array}{ccccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 3 & 0 \\ 0 & 0 & 13 & 8 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 4 & 1 & 0 & -4 \\ 1 & -2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

So we claim that this is a new basis of $\ker(A)$:

$$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -4 \\ 3 \\ 1 \end{pmatrix} \right\}$$

Note that this isn't the same as $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, but

$$\begin{aligned} \vec{v}_2 &= \vec{w}_1; & \vec{v}_1 &= 2\vec{w}_1 + \vec{w}_2; \\ \vec{v}_1 - 2\vec{v}_2 &= \vec{w}_2; & \vec{v}_2 &= \vec{w}_1; \\ 3\vec{v}_2 + \vec{v}_3 &= \vec{w}_3; & \vec{v}_3 &= -3\vec{w}_1 + \vec{w}_3. \end{aligned}$$

This gives what's called the *change of basis matrix*:

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} = \begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{pmatrix} \begin{pmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 3 \\ 0 & 0 & 1 \end{pmatrix};$$

Dual to finding a basis of the kernel, we can find a basis of the *image* of a matrix, $\mathrm{im}(A)$. The image of A is the span of the column vectors of A.

If A is an $m \times n$ matrix, then A eats a vector of $\mathbf{R}^n$, and it poops a vector of $\mathbf{R}^m$. The kernel of A is thus a subspace of $\mathbf{R}^n$, and the image of A is a subspace of $\mathbf{R}^m$.

The way we *compute* a basis of the image is not wildly different from the way in which we compute a basis of the kernel, but the operations are dual, and that can get confusing. To clear up our confusion, we’ll need some theorems!!

Theorem (Rank-Nullity Theorem)**.** *If A is an $m \times n$ matrix, then*

$$\dim(\ker(A)) + \dim(\operatorname{im}(A)) = n.$$

We are going to spend some quality time with this result.