18.06.14: 'Rank-nullity'

Lecturer: Barwick

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Theorem (Rank-Nullity Theorem). If A is an $m \times n$ matrix, then

 $\dim(\ker(A)) + \dim(\operatorname{im}(A)) = n.$



Suppose A were in reduced row echelon form:

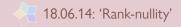
Then, we have a few columns (1, 3, 4, 5 in this case) that are distinct standard basis vectors, and the other columns can be written as a linear combination of these. So the rank here is 4.



When we have a vector \vec{x} with $A\vec{x} = \vec{0}$, we can write x_1, x_3, x_4 , and x_5 in terms of the other variables:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 0 \\ -7 \\ -6 \\ -7 \\ 1 \end{pmatrix}.$$

And as we know, 4 + 2 = 6.



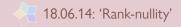
This pattern works in general. A matrix *A* is in rref iff it looks like this:

$$\left(\vec{A}^1 \quad \cdots \quad \vec{A}^{k_1-1} \quad \hat{e}_1 \quad \vec{A}^{1+k_1} \quad \cdots \quad \vec{A}^{k_2-1} \quad \hat{e}_2 \quad \cdots \quad \hat{e}_r \quad \vec{A}^{1+k_r} \quad \cdots \quad \vec{A}^n \right).$$

where the column vectors $\vec{A}^{1+k_i}, \ldots, \vec{A}_{k_{i+1}-1}$ all lie in the span of $\hat{e}_1, \ldots, \hat{e}_i$, but not in the span of $\hat{e}_1, \ldots, \hat{e}_{i-1}$.

The image is thus the span of the of distinct \hat{e}_i 's that appear:

 $r = \dim(\operatorname{im}(A)).$



When you compute the kernel, on the other hand, you express $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$ in terms of the other x_i 's. That is, you write any $\vec{x} \in \text{ker}(A)$ as a linear combination of vectors

$$\vec{v}_1, \ldots, \vec{v}_{k_1-1}, \vec{v}_{1+k_1}, \ldots, \vec{v}_{k_2-1}, \ldots, \vec{v}_{1+k_r}, \ldots, \vec{v}_n,$$

where each \vec{v}_j has a 1 in the *j*th spot, *something* in spots k_1, k_2, \ldots, k_r , and a 0 in every other spot.

In particular, these vectors must be linearly independent, so they form a basis of the kernel! And since there are n - r of them, we have

 $\dim(\ker(A)) = n - r,$



as desired.

This proves the Rank-Nullity Theorem when A is in rref!



But what if A isn't in rref? What then?

Well, we know we can perform row operations to get *A* into rref. (This is the magic of Gaussian elimination.)

row operations: $A \dashrightarrow MA$.

There's an invertible $m \times m$ matrix M such that MA is in rref.

Now we first recall that row operations don't change the kernel:

 $\ker(A) = \ker(MA).$



However, row operations absolutely *do* change the image:

 $\operatorname{im}(A) \neq \operatorname{im}(MA).$



Well, that's too bad! So what do we do to get out of trouble? We remember that the Rank-Nullity theorem only involves the *dimension* of the image, not the image itself.

And good news: *row operations don't change the dimension of the image*. So even though

 $\operatorname{im}(A) \neq \operatorname{im}(MA),$

we still have

 $\dim(\operatorname{im}(A)) = \dim(\operatorname{im}(MA)).$

(Why???)



Now, victory is ours! For any old $m \times n$ matrix A, we multiply on the left by an invertible $m \times m$ matrix M to get a matrix MA in rref, and we have

 $\dim(\ker(A)) + \dim(\operatorname{im}(A)) = \dim(\ker(MA)) + \dim(\operatorname{im}(MA)) = n.$

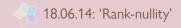
{*mic drop*}



Let's take a moment to imagine how our proof might have been different if we'd used column operations to get our matrix into rcef:

 $column \ operations: A \dashrightarrow AN,$

where *N* is an invertible $n \times n$ matrix.



The point here is that *column operations don't change the image*:

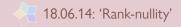
 $\operatorname{im}(A) = \operatorname{im}(AN).$

However, column operations absolutely *do* change the kernel:

 $\ker(A) \neq \ker(AN).$

BUT, column operations don't change the **dimension** of the kernel:

 $\dim(\ker(A)) = \dim(\ker(AN)).$



Using pure thought, tell me what the rank and nullity are of these matrices:

$$\left(\begin{array}{rrr} 5 & -15 \\ -2 & 6 \end{array}\right)$$

$$\left(\begin{array}{rrr}2&4&-138\\5&1&75\end{array}\right)$$



$$\left(\begin{array}{rrrr} 9 & 9 & 9 \\ 1 & 1 & 1 \\ 4 & 4 & 4 \end{array}\right)$$

$$\left(\begin{array}{rrrr}1 & 5 & 7\\ -2 & 6 & 3\\ -1 & 11 & 10\end{array}\right)$$



$$\left(\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 16 \end{array}\right)$$