## 18.06.16: The four fundamental subspaces

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Here it is again:

**Theorem** (Rank-Nullity Theorem). If A is an  $m \times n$  matrix, then

 $\dim(\ker(A)) + \dim(\operatorname{im}(A)) = n.$ 

We saw a proof of this by reducing to rref or rcef, and then checking it there. There's just one thing that might bug us here. If I think of the linear map

$$T_A : \mathbf{R}^n \longrightarrow \mathbf{R}^m$$
,

then we see that ker(A) is a subspace of the source  $\mathbb{R}^n$ , but im(A) is a subspace of the target  $\mathbb{R}^m$ . So why should these spaces be related?

To answer this question, there's another matrix we can contemplate, the transpose  $A^{T}$ . This is an  $n \times m$  matrix, and so it corresponds to a linear map in the other direction:

$$T_{A^{\mathsf{T}}} \colon \mathbf{R}^m \longrightarrow \mathbf{R}^n.$$

This is the map that takes a column vector  $\vec{v}$  and builds the column vector  $A^{\mathsf{T}}\vec{v}$ , but we can perform a trick here. Instead of thinking about transposing *A*, we can think about transposing the *vectors*.

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So  $(\mathbf{R}^m)^{\vee}$  will be the set of all *m*-dimensional *row vectors*; equivalently, the set of all  $1 \times m$  matrices; equivalently again, the set of all transposes

$$v := (\vec{v})^{\mathsf{T}}$$

of vectors  $\vec{v} \in \mathbf{R}^m$ . We call  $(\mathbf{R}^m)^{\vee}$  the *dual*  $\mathbf{R}^m$ .

So, e.g., if 
$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$
, then  $\vec{v} = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \end{pmatrix}$ .

The neat thing about row vectors is that they *do stuff* to column vectors. If  $\vec{v} \in \mathbf{R}^n$  and  $\vec{w} \in (\mathbf{R}^n)^{\vee}$ , then  $\vec{w}\vec{v}$  is a *number*. (Question: How is this related to the dot product?)

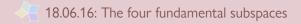
If  $V \subseteq \mathbf{R}^n$  is a vector subspace, then we define

$$V^{\perp} \coloneqq \{ \underline{w} \in (\mathbf{R}^n)^{\vee} \mid \text{for any } \vec{v} \in V, \ \underline{w}\vec{v} = 0 \} \subseteq (\mathbf{R}^n)^{\vee}.$$

Dually, if  $W \subseteq (\mathbf{R}^n)^{\vee}$  is a vector subspace, then we define

$$W^{\perp} := \{ \vec{v} \in \mathbf{R}^n \mid \text{for any } \vec{w} \in W, \ \vec{w}\vec{v} = 0 \} \subseteq \mathbf{R}^n.$$

Fact: dim(V) =  $n - \dim(V^{\perp})$ , and dim(W) =  $n - \dim(W^{\perp})$ . (Why?)



Now, since

$$(A^{\mathsf{T}}\vec{v})^{\mathsf{T}} = (\vec{v})^{\mathsf{T}}A = \vec{v}A,$$

we can leave A just as it is, and we can consider the linear map

 $T_A^{\vee}: (\mathbf{R}^m)^{\vee} \longrightarrow (\mathbf{R}^n)^{\vee}$ 

given by the formula

$$T_A^{\vee}(\underline{v}) \coloneqq \underline{v}A.$$

So when we contemplate the kernel and image of  $A^{\mathsf{T}}$ , we can think about it via the map  $T_A^{\lor}$ .

For example,  $\ker(A^{\mathsf{T}})$  is the set of all vectors  $\underline{v} \in (\mathbb{R}^n)^{\vee}$  such that  $\underline{v}A = \underline{0}$ . This space is also called the *cokernel* or the *left kernel* of A. I write coker(A).

We also have the image of  $A^{\mathsf{T}}$ , which is the set of all row vectors  $\vec{w} \in (\mathbb{R}^n)^{\vee}$  such that there exists a row vector  $\vec{v} \in (\mathbb{R}^m)^{\vee}$  for which  $\vec{w} = \vec{v}A$ . This space is also called the *coimage* of *A*, or, since its the span of the columns of  $A^{\mathsf{T}}$ , which is the span of the rows of *A*, it is also called the *row space* of *A*. I write coim(*A*).



## In all, we have four vector spaces that are what Strang call the *fundamental subspaces* attached to *A*:

 $\operatorname{ker}(A)$ ,  $\operatorname{im}(A)$ ,  $\operatorname{coker}(A) \coloneqq \operatorname{ker}(A^{\mathsf{T}})$ ,  $\operatorname{coim}(A) \coloneqq \operatorname{im}(A^{\mathsf{T}})$ .

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## Here's the abstract statement of the Rank-Nullity Theorem:

(1)  $\ker(A) = \operatorname{coim}(A)^{\perp}$ , so that

$$\dim(\ker(A)) = n - \dim(\operatorname{coim}(A)).$$

(2)  $\operatorname{im}(A) = \operatorname{coker}(A)^{\perp}$ , so that

 $\dim(\operatorname{im}(A)) = m - \dim(\operatorname{coker}(A)).$ 

(3) A provides a bijection  $\operatorname{coim}(A) \cong \operatorname{im}(A)$ , so that

 $\dim(\operatorname{coim}(A)) = \dim(\operatorname{im}(A)).$