18.06.22: Determinants

Lecturer: Barwick

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So if

$$A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$$

is a collection of n vectors in $\mathbf{R}^n$, taken in order – in other words, an $n \times n$ matrix – then we are trying to define a number

$$\det(A) = \det(\vec{v}_1, \ldots, \vec{v}_n) \in \mathbf{R}$$

that is meant to measure the *signed n-dimensional volume* of the parallelopiped spanned by $\vec{v}_1, \ldots, \vec{v}_n$.

Note that $\vec{v}_i = A\hat{e}_i$. So $\det(A)$ can also be thought of as the signed volume of the unit cube after being modified by the matrix A.

Question. One of the only things we know about the determinant is that it's supposed to be the case that $\det(A) \neq 0$ if and only if A is invertible. What has that to do with the signed volume?

We will focus on making a wish list of properties for the determinant. We will end up writing enough of these down to describe it for any matrix uniquely. Here are the first two:

(1) *Normalization.* The identity matrix has determinant 1:

$$\det(I) = \det\left(\hat{e}_1, \cdots, \hat{e}_n\right) = 1.$$

(2) *Scaling.* For any real number $r \in R$,

$$\det\left(\vec{v}_1, \cdots, \vec{v}_{i-1}, r\vec{v}_i, \vec{v}_{i+1}, \cdots, \vec{v}_n\right) = r \det\left(\vec{v}_1, \cdots, \vec{v}_n\right).$$

Question. Suppose $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$. Using only what we already know, what is

$$\det \operatorname{diag}(\lambda_1, \ldots, \lambda_n) = \det\left(\lambda_1 \hat{e}_1, \cdots, \lambda_n \hat{e}_n\right)?$$

We can elaborate our second condition a bit by thinking about addition. What should happen here:

$$\det\left(\vec{v}_1, \cdots, \vec{v}_{i-1}, \vec{v}_i + \vec{w}_i, \vec{v}_{i+1}, \cdots, \vec{v}_n\right)?$$

We thus have a new version of our second criterion:

(2) *Multilinearity.* For any real numbers $r, s \in \mathbf{R}$,

$$\begin{aligned}\det(\vec{v}_1, \cdots, r\vec{x}_i + s\vec{y}_i, \cdots, \vec{v}_n) &= r\det(\vec{v}_1, \cdots, \vec{x}_i, \cdots, \vec{v}_n)\\ &\quad + s\det(\vec{v}_1, \cdots, \vec{y}_i, \cdots, \vec{v}_n).\end{aligned}$$

Now let's impelement this idea that the n-dimensional volume of something lower dimensional should be zero. (The length of a point, the area of a line, the volume of a square, etc., should all be zero.)

So, here's the third condition.

(3) *Alternation.* The determinant

$$\det(\vec{v}_1, \cdots, \cdots, \vec{v}_n) = 0$$

if any two of the $\vec{v}_i$s are equal.

Question. That word alternation seems odd here. It seems like alternation should have to do with what happens to the determinant when we swap to columns. So, what does happen to the determinant? How does this relate to the idea of signed volume??

Question. So we've seen what happens to the determinant if you multiply a column by a number and if you swap two columns. What's left of the column operations? What happens then?

Amazingly, this wishlist *already* uniquely identifies the determinant: there is only one map det: $\mathbf{Mat}_n(\mathbf{R}) \longrightarrow \mathbf{R}$ that is

(1) normalized, so that $\det(I) = 1$;

(2) multilinear, so that $\det\left(\vec{v}_1, \cdots, r\vec{x}_i + s\vec{y}_i, \cdots, \vec{v}_n\right) = \det\left(\vec{v}_1, \cdots, \vec{x}_i, \cdots, \vec{v}_n\right) + s\det\left(\vec{v}_1, \cdots, \vec{y}_i, \cdots, \vec{v}_n\right)$; and

(3) alternating, so that $\det\left(\vec{v}_1, \cdots, \cdots, \vec{v}_n\right) = 0$ if any two of the $\vec{v}_i$s are equal.

(Moral: Instead of studying individual numbers, study the whole map, and understand how it is transformed.)

Now you might be unhappy with this description, because it doesn't directly provide a formula. But, along the way, we saw how the determinant transforms under column operations. So if we start with an $n \times n$ matrix A, we can perform column operations to get an easier matrix.

Along the way, we keep track of the operations we've done, and we have the following rules:

(1) Multiplying a column by a number multiplies the determinant by that number.

(2) Swapping two columns changes the sign.

(3) Adding a multiple of another column to a given column doesn't change the determinant.

Let's compute the determinant of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

using column operations.

Let's think about this still more: if we perform column operations to the identity matrix I, then we can get *any* invertible matrix we want. The sequence of column operations gives us a computation for $\det(N)$.

Now if A is any $n \times n$ matrix, then when we use those same column operations, in that same order, to A, we get AN, and what we've learned is that

$$\det(AN) = \det(A)\det(N).$$