18.06.28: Complex vector spaces

I worked so hard to understand it that it must be true.
— James Richardson
From last time …

I was alluding to a way to make complex multiplication easier to understand. The idea is this: for any $z = a + bi \in \mathbb{C}$, you may consider the matrix

$$M_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$ 

On the newest problem set, you’ll show that addition of complex numbers is addition of these matrices, multiplication of complex numbers is multiplication of these matrices (!), and one more thing …
Complex conjugation is the map $z \mapsto \bar{z}$ that carries $z = a + bi$ to $\bar{z} = a - bi$. You’ll see that $M_{\bar{z}} = M_{\bar{z}}^T$.

Complex conjugation can be used to extract the real and complex parts of your complex number:

\[
2a = z + \bar{z};
\]
\[
2bi = z - \bar{z}.
\]
Complex conjugation also gives you the length of the vector $\vec{v} \in \mathbb{R}^2$ corresponding to $z \in \mathbb{C}$:

$$\|\vec{v}\|^2 = z\overline{z}.$$  

So

$$z = \sqrt{z\overline{z}} \exp(i\theta)$$

for some (and hence infinitely many) $\theta \in \mathbb{R}$.

If $z \neq 0$, there is a unique such $\theta \in [0, 2\pi)$; this is sometimes called the \textit{argument} of $z$, but it’s annoying to write down a good formula. It’s better to think of it as an element of $\mathbb{R}/2\pi\mathbb{Z}$.  

One last general thing about the complex numbers, just because it’s so important.

**Theorem** ("Fundamental theorem of algebra"). *For any polynomial*

\[ f(z) = \sum_{i=0}^{n} \alpha_i z^i \]

*with complex coefficients \( \alpha_i \) such that \( \alpha_n \neq 0 \), there exist complex numbers \( w_1, \ldots, w_n \) such that*

\[ f(z) = \alpha_n (z - w_1)(z - w_2) \cdots (z - w_n). \]

This is actually a theorem of *topology*, not algebra, but there you go.
Here’s a cool example: $f(z) = z^n - 1$. Let’s find the roots!
The set $\mathbb{C}^n$ is the set of column vectors

\[ \mathbf{v} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \]

with $z_i \in \mathbb{C}$. One can add such vectors componentwise, and one can multiply any such vector with a complex scalar.

So this is the fundamental example of a complex vector space.
Note that $\mathbb{R}^n \subset \mathbb{C}^n$. Now if $v \in \mathbb{C}^n$, then $v = \bar{v}$ if and only if $v \in \mathbb{R}^n$. 
More generally, a complex vector subspace $V \subseteq \mathbb{C}^n$ is a subset such that:

(1) for any $v, w \in V$, one has $v + w \in V$;

(2) for any $v \in V$ and any $z \in \mathbb{C}$, one has $zv \in V$.  

Vectors $v_1, \ldots, v_k$ span a vector subspace $V \subseteq \mathbb{C}^n$ over $\mathbb{C}$ if and only if every vector $w \in V$ can be written as a $\mathbb{C}$-linear combination of the $v_i$, i.e.,

$$w = \sum_{i=1}^{k} z_i v_i,$$

where each $z_i \in \mathbb{C}$.
Similarly, the vectors \( v_1, \ldots, v_k \) are *linearly independent over \( \mathbb{C} \) if and only if any vanishing \( \mathbb{C} \)-linear combination

\[
\sum_{i=1}^{k} z_i v_i = 0
\]

is a trivial \( \mathbb{C} \)-linear combination, so that \( z_1 = \cdots = z_k = 0 \).

A \textit{\( \mathbb{C} \)-basis} of \( V \) is thus a collection of vectors of \( V \) that is linearly independent over \( \mathbb{C} \) and spans \( V \) over \( \mathbb{C} \).
Let’s do an example to appreciate the distinction. Let’s think of $\mathbb{C}^2$, and let’s think of the complex line

$$L = \left\{ \left( \begin{array}{c} z \\ w \end{array} \right) \in \mathbb{C}^2 \mid 3z - 2w = 0 \right\} \subset \mathbb{C}^2.$$ 

Now $\mathbb{C}^2 \cong \mathbb{R}^4$, so that complex line is a real plane:

$$L = \left\{ \left( \begin{array}{c} z_1 \\ z_2 \\ w_1 \\ w_2 \end{array} \right) \in \mathbb{R}^4 \mid \begin{array}{c} 3z_1 - 2w_1 = 0 \\ 3z_2 - 2w_2 = 0 \end{array} \right\} \subset \mathbb{R}^4.$$
The single vector \( \begin{pmatrix} 2 \\ 3 \end{pmatrix} \) forms a \( \mathbb{C} \)-basis of \( L \).

Another legit \( \mathbb{C} \)-basis would be the single vector \( \begin{pmatrix} 2i \\ 3i \end{pmatrix} \).

The vectors
\[
\left\{ \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 3 \end{pmatrix} \right\}
\]
forms an \( \mathbb{R} \)-basis of \( L \) over \( \mathbb{R} \).
Another example: consider the complex vector subspace $W \subset \mathbb{C}^2$ spanned by \[
\begin{pmatrix} i \\ 1 \end{pmatrix}.
\]

Here's an important sentence to parse correctly: *W does not have a \(\mathbb{C}\)-basis consisting of real vectors.*

A real basis for $W \subset \mathbb{R}^4$ consists of \[
\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]
Proposition. Any complex vector subspace $W \subset \mathbb{C}^n$ of complex dimension $k$ has an underlying real vector space of dimension $2k$.

To see why, take a $\mathbb{C}$-basis $\{w_1, \ldots, w_k\}$ of $W$. Now $\{w_1, iw_1, \ldots, w_k, iw_k\}$ is an $\mathbb{R}$-basis of $W$. 
In the other direction, a real vector subspace \( V \subseteq \mathbb{R}^n \) generates a complex vector subspace \( V_C \subseteq \mathbb{C}^n \), called the *complexification*; this is the set of all \( \mathbb{C} \)-linear combinations of elements of \( V \):

\[
V_C := \left\{ w \in \mathbb{C}^n \mid w = \sum_{i=1}^{k} \alpha_i v_i, \text{ for some } \alpha_1, \ldots, \alpha_k \in \mathbb{C}, v_1, \ldots, v_k \in V \right\}.
\]

Note that not all complex vector subspaces of \( \mathbb{C}^n \) are themselves complexifications; the complex vector subspace \( W \subset \mathbb{C}^2 \) spanned by \( \begin{pmatrix} i \\ 1 \end{pmatrix} \) provides a counterexample. (A complex vector space is a complexification if and only if it has a \( \mathbb{C} \)-basis consisting of real vectors.)
Now, most importantly, we may speak of complex matrices (i.e., matrices with complex entries).

*All the algebra we’ve done with matrices over $\mathbb{R}$ works perfectly for matrices over $\mathbb{C}$, without change.*
However, the freedom to contemplate complex matrices offers us new horizons when it comes to questions about eigenspaces and diagonalization. Let’s contemplate the matrix

\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

The characteristic polynomial \( p_A(t) = t^2 + 1 \) doesn’t have any real roots, so there’s no hope of diagonalizing \( A \) over \( \mathbb{R} \).

Over \( \mathbb{C} \), however, we find eigenvalues \( i, -i \). Let’s try to diagonalize \( A \).
Let’s begin with $L_i = \ker(iI - A) = \ker\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$. It’s dimension 1, and it’s spanned by the vector $\begin{pmatrix} 1 \\ -i \end{pmatrix}$.

And $L_{-i} = \ker\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}$ is dimension 1 and spanned by $\begin{pmatrix} 1 \\ i \end{pmatrix}$. 
Note that neither $L_i$ nor $L_{-i}$ is a complexification. However, we do have a basis
\[ \left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix} \right\} \] of $\mathbb{C}^2$ consisting of eigenvectors of $A$, and writing $T_A$ in terms of this basis gives us the matrix
\[
\begin{pmatrix}
 i & 0 \\
 0 & -i
\end{pmatrix}.
\]

So $A$ is not diagonalizable over $\mathbb{R}$, but it is diagonalizable over $\mathbb{C}$. 
There’s one more new thing you can do with a complex matrices that doesn’t quite work for real matrices: you can conjugate their entries. Of particular import is the conjugate transpose:

\[ A^* := (\overline{A})^T = (\overline{A^T}). \]

We’ll understand the significance of this operation next time.