# Solutions to the practice final exam 18.06

## Problem 1

- True
- True
- False: consider the similar matrices $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.
- True
- False: the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has both of its eigenvalues equal to zero.
- False: the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is again a counterexample.

## Problem 2

- We rewrite this system in the matrix form

$$\begin{pmatrix} 1 & 3 & 5 \\ 1 & 2 & 2 \\ 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

  Let us denote the matrix $A$. The row reduced form of $A$ is $\begin{pmatrix} 1 & 3 & 5 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$. There are two pivots, thus, the kernel of $A$ is one-dimensional and $z$ is the free variable. We find that the basis of the kernal is $\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$. Therefore, the general solution of the homogeneous system is $s \cdot \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$, for $s \in \mathbb{R}$.

- Let $a = 0$, $b = -1$, and $c = -2$. The special solution is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Thus, the general solution is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$, for $s \in \mathbb{R}$.

- Consider $a = 0$, $b = 0$, and $c = -2$. Since the last equation is (2*second equation - first equation) we conclude that if $a = 0$ and $b = 0$, then for a solution to exist $c$ must equal to 0.

## Problem 3

This matrix is in block-diagonal form. The characteristic polynomial is $(t-1)(t-2)(t-k)$. Now, if $k \neq 1, 2$ this matrix is DZ over $\mathbb{C}$, since all its eigenvalue are distinct (if $k \in \mathbb{R}$ it is DZ over $\mathbb{R}$ as well).

Let $k = 1$. The corresponding eigenspace is the kernel of

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

which is 2−dimensional. Thus, the matrix is DZ over $\mathbb{R}$ and $\mathbb{C}$.

Finally, let $k = 2$. The corresponding eigenspace is the kernel of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix},$$

which is 1−dimensional. Thus, the matrix is not DZ over $\mathbb{C}$ and $\mathbb{R}$.

## Problem 4

We will do row reduction. Substract the 4th row from the 5th one, the 3d row from the 4th one and so on. This operations do not change the determniant and we end up with matrix

$$\tilde{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \det(\tilde{A}) = \det(A).$$

No we will perform 4 swaps to get

$$\tilde{\tilde{A}} = \begin{pmatrix} 5 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \det(\tilde{\tilde{A}}) = (-1)^4 \det(A).$$

Determinant of this matrix is 5. Therefore, $\det(A) = 5$ as well.

## Problem 5

Note that this matrix is in block-upper-triangular form where the first matrix is Hermitian and the second block is 1-by-1. Thus, all the eigenvalues of this matrix are real. Its characteristic polynomial is $p(t) = (t^3 - 3t^2 - 16t - 12)(t-1)$. By trial and error we find that $p(t) = (t-6)(t+2)(t+1)(t-1)$. Therefore, all the eigenvalues are distinct, so the matrix is DZ over $\mathbb{C}$ and all the eigenspaces are 1-dimensional. The eigenvectors are $(1, -21i, 6-9i, 13, 0)^T, (1+3i, -2-i, 5, 0)^T, (-1, 1+2i, 1, 0)^T, (-13+24i, 11-8i, 7-36i, 30)$.

The determinant of $A^3 + 2A$ is the product of its eigenvalues which are $6^3 + 2 \cdot 6, (-2)^3 + 2 \cdot (-2), (-1)^3 + 2 \cdot (-1), 1^3 + 2 \cdot 1$.

## Problem 6

Coimage is the row space. In the row reduced form there are three pivots, so the row space is 3-dimensional. Therefore, the row space is the whole $\mathbb{R}^3$. Then the projection of $v$ is the vector itself.