18.06 Problem Set 2. Solutions

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Problem 1

Invert the following square matrices using whatever method you prefer.

Matrix	Inverse Matrix
1. $\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$,	1. $\frac{1}{21}\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$,
2. $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$,	2. $\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$,
3. $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 6 \\ 4 & 6 & 8 \end{pmatrix}$,	3. $\begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & 0 \end{pmatrix}$,
4. $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$,	4. $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$,
5. $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$,	5. $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$,
6. $\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{pmatrix}$.	6. $\begin{pmatrix} 25 & -300 & 1050 & -1400 & 630 \\ -300 & 4800 & -18900 & 26880 & -12600 \\ 1050 & -18900 & 79380 & -117600 & 56700 \\ -1400 & 26880 & -117600 & 179200 & -88200 \\ 630 & -12600 & 56700 & -88200 & 44100 \end{pmatrix}$.

Remarks:

1d) It is a permutation matrix P_π, where π is a permutation π of a set of 6 elements

$$\pi\colon \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\},$$

given in two-line form by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 4 & 6 & 2 \end{pmatrix}.$$

Multiplying a permutation matrix P_π times a column vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_6 \end{pmatrix}$ will permute the rows of x :

$$P_\pi \mathbf{x} = \begin{pmatrix} x_{\pi(1)} = x_3 \\ x_{\pi(2)} = x_5 \\ \dots \\ x_{\pi(6)} = x_2 \end{pmatrix}.$$

Thus, the inverse of P_π is $P_{\pi^{-1}} = P_\pi^T$, where P_π^T is the transpose of P_π.

1e) Note that

$$A\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 + x_2 \\ x_1 - x_2 + x_3 \\ x_4 - x_5 \\ x_5 \end{pmatrix}.$$

Thus, the inverse matrix A^{-1} should act in the following way:

$$A^{-1} \begin{pmatrix} x_1 \\ -x_1 + x_2 \\ x_1 - x_2 + x_3 \\ x_4 - x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

Therefore,

$$A^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + x_1 \\ x_3 + x_2 \\ x_4 + x_5 \\ x_5 \end{pmatrix}.$$

1f) The matrix H with the entries $H_{i,j} = \frac{1}{i+j-1}$ is called a Hilbert matrix. There is a general formula for computing the inverse of H:

$$(H^{-1})_{ij} = (-1)^{i+j}(i+j-1) \cdot \binom{n+i-1}{n-j} \cdot \binom{n+j-1}{n-i} \cdot \binom{i+j-2}{i-1}^2,$$

where n is the order of the matrix. In particular, it means that the entries of the inverse matrix are all integer.

Problem 2

The matrix is invertible if and only if its determinant is non-zero. The determinant of a 2×2 matrix

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

If the unique negative entry is a or d then $\det < 0$, otherwise $\det > 0$. In any case, the determinant is not equal to zero, so the matrix is invertible.

Problem 3

Let us look at 3×3 case first. Consider as an example

$$L_{1,3}(r)A = \begin{pmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} + r \cdot a_{31} & a_{12} + r \cdot a_{32} & a_{13} + r \cdot a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$AL_{1,3}(r) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} + r \cdot a_{11} \\ a_{21} & a_{22} & a_{23} + r \cdot a_{12} \\ a_{31} & a_{32} & a_{33} + r \cdot a_{13} \end{pmatrix}.$$

The above computation can be easily extended to the $n \times n$ case.

We see the pattern:

- Multiplication by $L_{i,j}(r)$ on the left results in adding r times j-th row vector to the i-th row vector while all the other row vectors stay the same;
- Multiplication by $L_{i,j}(r)$ on the right results in adding r times i-th column column to the j-th column vector while all the other row vectors stay the same.

Using the Gauss-Jordan method we see that

$$L_{ij}(r)^{-1} = L_{ij}(-r).$$

Problem 4

Let us divide this large 18×18 matrix, which we denote by A, into 9 blocks of 6×6 matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Note that A_{21}, A_{31} and A_{32} are zero matrices. So our matrix is block upper triangular.

The matrices A_{11}, A_{22}, A_{33} all are almost lower triangular, more precisely, they become lower triangular after two steps of the Gauss elimination process and the entries on the diagonal are non-zero. Then we see that $\det A_{11} = 1 \cdot 2 \cdot \dots \cdot 6$, $\det A_{22} = 7 \cdot 8 \cdot \dots \cdot 12$, $\det A_{11} = 13 \cdot 14 \cdot \dots \cdot 18$.

Recall that $\det A = \det A_{11} \cdot \det A_{22} \cdot \det A_{33} = 1 \cdot 2 \cdot \dots \cdot 18$. In particular, this matrix is invertible.

Problem 5

The first solution is the same one as for the third problem in the midterm. Let us show that the span of the columns of the matrix corresponding to this system of equations is $\mathbb{R}^{512}$. Denote the columns by $v_1, \dots, v_{512}$. observe that $\frac{1}{511} \sum_{i=1}^{512} v_i = (1, 1, \dots, 1)$, so for all i we have $\mathbf{e}_i = (\frac{1}{511} \sum_{i=1}^{512} v_i) - v_i$. Thus, all $\mathbf{e}_i$ lie in the span of $\{v_i\}$. Therefore, there exists a solution of this system of equations. Also, by two-out-of-three criterion the vectors $\{v_i\}$ are linearly independent, so the solution is unique.

There is also a way to solve this system of equations directly by an elimination procedure.