18.06 PSet 3 Solution

Problem 1

(a) A is invertible, so $\ker A = \{0\}$.

(b), (c) and so on:

Let us solve the $n \times n$ case for $n \geq 3$. Let c_j be the j-th column of our matrix A. A direct computation shows that

$$c_2 - c_1 = c_3 - c_2 = \cdots = c_n - c_{n-1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

hence $c_1 - 2c_2 + c_3 = c_2 - 2c_3 + c_4 = \cdots = c_{n-2} - 2c_{n-1} + c_n = 0$. From these linear relations between columns of A, we can extract $n-2$ solutions to the equation $A\vec{x} = \vec{0}$. They are

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \vec{x}_{n-2} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

It is not hard to check that $\vec{x}_1, \ldots, \vec{x}_{n-2}$ are linearly independent vectors, so

$$\dim(\ker A) \geq n - 2. \tag{1}$$

On the other hand, notice that the first 2 columns of A are linearly independent, therefore $\dim(\text{im } A) \geq 2$. By the Rank-Nullity Theorem (see Lecture 13 slides), we know that

$$\dim(\ker A) = n - \dim(\text{im } A) \leq n - 2. \tag{2}$$

Compare (1) and (2) we conclude that $\dim(\ker A) = n - 2$ and therefore $\{\vec{x}_1, \ldots, \vec{x}_{n-2}\}$ forms a basis of $\ker A$.

Remark 0.1. What described above is exactly the column operation method covered in Lecture 13.

Problem 2

Both the method and answer are identical to Problem 1.

Problem 3

If we write down the matrix A, it is easy to find that

$$c_1 = c_3 = c_5 = \cdots = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{pmatrix}$$

and

$$c_2 = c_4 = c_6 = \cdots = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}.$$

Use the argument in Problem 1, one conclude that

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

is a basis of $\ker A$. There are $(n-2)$ of them.

Problem 4

Suppose $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a vector in $\ker X$. It is equivalent to say that

$$\begin{cases} Av_1 + Bv_2 = 0 \\ Cv_2 = 0 \end{cases}.$$

Hence for any $v_1 \in \ker A$, the vector $\begin{pmatrix} v_1 \\ 0 \end{pmatrix}$ lives in $\ker X$, i.e.,

$$\begin{pmatrix} \ker A \\ 0 \end{pmatrix} \subset \ker X,$$

in particular

$$\dim \ker X \geq \dim \ker A.$$

On the other hand, for $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \ker X$, we know that $v_2 \in \ker C$. Moreover, fix $v_2 \in \ker C$, the solutions to

$$Av_1 = -Bv_2$$

is either the empty set or an affine space of dimension $\dim \ker A$ (meaning that the difference of any two solutions v_1 and v_1' is a vector in $\ker A$), so we conclude that

$$\dim \ker X \leq \dim \ker A + \dim \ker C.$$

Combining two parts, we see that

$$\dim \ker A \leq \dim \ker X \leq \dim \ker A + \dim \ker C. \tag{3}$$

Remark 0.2. Maybe a better way to reach the final result is to observe that

$$q + \text{column rank of } A \geq \text{column rank of } X \geq \text{column rank of } C + \text{column rank of } A,$$

and deduce the final result from the Rank-Nullity Theorem.

Problem 5

Direct computation shows that

$$Q^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

hence $Q^2 - Q - I = 0$, where I is the 2 by 2 identity matrix. Use the relation $Q^2 = Q + I$ repetitively, we see that

$$\begin{aligned}
Q^{-1} &= Q^{-1} \cdot I = Q^{-1}(Q^2 - Q) = Q - I = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \\
Q^{-2} &= (Q^{-1})^2 = (Q - I)^2 = Q^2 - 2Q + I = 2I - Q = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \\
Q^{-3} &= Q^{-2}Q^{-1} = (2I - Q)(Q - I) = -Q^2 + 3Q - 2I = 2Q - 3I = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}, \\
Q^3 &= Q^2Q = (Q + I)Q = Q^2 + Q = 2Q + I = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}.
\end{aligned}$$

In general, we have

$$Q^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$$

and we can deduce this by induction on n, using the fact that

$$Q^n = Q^{n-2}Q^2 = Q^{n-2}(Q + I) = Q^{n-1} + Q^{n-2}.$$

Taking the determinant of Q^n, we see that

$$f_{n+1}f_{n-1} - f_n^2 = \det Q^n = (\det Q)^n = (-1)^n,$$

therefore

$$f_n^2 + (-1)^n = f_{n-1}f_{n+1}.$$

There are many things we can do by playing with Q. For instance, consider

$$Q^{n+m} = Q^nQ^m = Q^mQ^n,$$

then we can write down

$$\begin{pmatrix} f_{n+m+1} & f_{n+m} \\ f_{n+m} & f_{n+m-1} \end{pmatrix} = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \begin{pmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{pmatrix}.$$

Compare the $(1,1)$-entry, we see that

$$f_{n+m+1} = f_{n+1}f_{m+1} + f_n f_m.$$

In particular, if we choose $n = m$, we see that

$$f_{2n+1} = f_{n+1}^2 + f_n^2.$$

Therefore f_{2n+1} can be expressed as a sum of two squares, hence (if you know some number theory) f_{2n+1} is not a multiple of 3, 7, 11, 19, 23,. . . .

Problem 6

Let c_j be the j-th column of our matrix F. By the definition of Fibonacci numbers, we see that

$$c_1 + c_2 - c_3 = c_2 + c_3 - c_4 = \cdots = c_{n-2} + c_{n-1} - c_n = 0.$$

Use the method in Problem 1, we find that a basis of $\ker F$ is given by

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

There are $(n-2)$ of them.