MIT 18.06 Advanced Standing Exam

Your name is: ______________________________

August 2016

problem	**score**
1	
2	
3	
4	
5	
6	
total	

Problem 1:

Suppose A is a 10×10 real-symmetric matrix with eigenvalues 1,2,3,4,5,6,7,8,9,10, and corresponding (real) eigenvectors are $x_1, \ldots, x_{10}$. You are solving the equation $(A - \mu I)x = b$, for some number μ, and you notice that the solution x is blowing up ($\|x\| \to \infty$) as μ approaches some number μ_0.

(a) What are the possible values of μ_0 for which this could be true?

(b) If $\mu = 1.001$, give a good choice of approximate formula for x in terms of b and *one* of the eigenvectors of A, assuming b is chosen at random.

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Problem 2:

The complete solution to $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + d \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

for any arbitrary constants c and d.

(a) If A is an $m \times n$ matrix with rank r, give as much true information as possible about the integers m, n, and r.

(b) Construct an explicit example of a possible matrix A and a possible right-hand side $\vec{b}$ with the solution $\vec{x}$ above. (There are many acceptable answers; you only have to provide one.)

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Problem 3:

A sequence of numbers $f_0, f_1, f_2, \ldots$ is defined by the recurrence

$$f_{k+2} = 4f_{k+1} + 3f_k,$$

with starting values $f_0 = 1$, $f_1 = 1$. (Thus, the first few terms in the sequence are $1, 1, 7, 31, 145, 673, 3127, \ldots$.)

(a) Defining $\vec{u}_k = \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix}$, re-express the above recurrence as $\vec{u}_{k+1} = A\vec{u}_k$, and give the matrix A.

(b) Find the eigenvalues of A, and use these to predict what the ratio f_{k+1}/f_k of successive terms in the sequence will approach for large k.

(c) The sequence above starts with $f_0 = f_1 = 1$, and $|f_k|$ grows rapidly with k. Keep $f_0 = 1$, but give a *different* value of f_1 that will make the sequence (with the *same recurrence* $f_{k+2} = 4f_{k+1} + 3f_k$) approach *zero* ($f_k \to 0$) as $k \to \infty$.

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Problem 4:

True or false. Give a counter-example if *false*. (You *don't* need to provide a reason or proof if true.)

(a) If Q is an orthogonal matrix, then $|\det Q| = 1$.

(b) If A is a Markov matrix, then $d\vec{u}/dt = A\vec{u}$ approaches some finite constant vector (a "steady state") for any initial condition $\vec{u}(0)$.

(c) If S and T are subspaces of $\mathbb{R}^3$, then their intersection (points in *both* S *and* T) is also a subspace.

(d) If S and T are subspaces of $\mathbb{R}^3$, then their union (points in *either* S *or* T) is also a subspace.

(e) The column space of AB contains the column space of A.

(f) The column space of AB is contained in the column space of A.

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Problem 5:

Recall that A and A^T have the same eigenvalues for a square matrix A.

(a) Let x be an eigenvector of A for an eigenvalue λ_x, and let y be an eigenvector of A^T for an eigenvalue λ_y. Show that x and y are orthogonal ($y^T x = 0$) if $\lambda_x \neq \lambda_y$. (Do *not* assume A is symmetric!)

(b) If $A = \begin{pmatrix} 1 & 1 \\ \epsilon & 1 \end{pmatrix}$ for some number $\epsilon < 1$, then its eigenvalues are $\lambda_\pm = 1 \pm \sqrt{\epsilon}$ and the corresponding eigenvectors are $x_\pm = \begin{pmatrix} 1 \\ \pm\sqrt{\epsilon} \end{pmatrix}$. Find the eigenvectors $y_\pm$ of A^T, and check that $y_-^T x_+ = 0$, consistent with what you showed in the previous part.

(c) For $\epsilon = 0$, then $x_+ = x_-$ and the matrix A is ______________. In this case, $y_+^T x_+ =$ ____.

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Problem 6:

Your 18.06 TA has a matrix A and wants you to solve $A\vec{x} = \vec{b}$. Being a sadist like most TAs, however, she does not tell you what A is, and instead gives you a matrix $B = XA$. All you are told about X is that it is an invertible matrix.

(a) Why are you being asked this question? Specifically, what technique in the 18.06 is the transformation $A \to B$ most similar to: (i) Gaussian elimination (ii) Gram-Schmidt; (iii) diagonalization; (iv) similar matrices; or (v) least-squares? Why?

(b) Which of the four fundamental subspaces of A, if any, can you determine from B alone?

(c) What property/properties *must* B have in order for $A\vec{x} = \vec{b}$ to have a solution for *any* $\vec{b}$?

(d) Suppose $B = \begin{pmatrix} 1 & 2 & -1 \\ 4 & 9 & -6 \\ 1 & 4 & -5 \end{pmatrix}$ and *one* solution is $\vec{x} = (1, 2, 3)^T$. What is $X\vec{b}$?

(e) Using the the information from part (d), find *all* solutions $\vec{x}$.

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