Chutes-and-Ladders

September 7, 2017

```
In [1]: using PyPlot, Interact
```

1 Chutes and Ladders

Chutes and Ladders, a version of an ancient Indian board game called Snakes and Ladders, is a simple and popular children's board game.

- There are 100 numbered spaces, plus an unmarked starting position 0.
- Players take turns generating a random number from 1 to 6 (e.g. by rolling a die or spinning a wheel), and move a marker that many spaces.
- If you land at the bottom of a ladder or the top of a chute (snake), then your marker is transported across the board up the ladder or down the chute.
- The first player whose marker reaches position 100 wins.

Here is an image of a game board:

A simple question that one might ask is: **how many moves does it typically take to finish the game**?

It turns out that an elegant analysis of this game is possible via Markov matrices. Reviews of this idea can be found in this 2011 blog post or this article by Jeffrey Humpherys at BYU.

The key idea is to represent the board by a 101×101 matrix M, whose entry $M_{i,j}$ is the **probability of going from position j to position i**.

1.1 Simplified game: No chutes or ladders

To start with, let's analyze a boring version of the game, in which there are no chutes or ladders. On each turn, you simply move forward 1 to 6 spaces until you reach the end.

The corresponding matrix M is essentially:

$$M_{i,j} = \begin{cases} \frac{1}{6} & j \in \{i-1, i-2, \ldots, i-6\} \\ 0 & \text{otherwise} \end{cases}$$

since there is a 1/6 chance of moving 1,2,...,6 spaces from j. However, the final row is modified, because you can get to space 100 from space 99 if you roll anything, from space 98 if you roll a 2 or more, etcetera. And once you get to position 100, you stay there.

```
In [2]: M = zeros(101,101)
        for i = 2:100
            M[i,max(1,i-6):(i-1)] = 1/6
        end
        # last row
        for i = 1:6
            M[101,101-i] = (7-i)/6 # = 6/6, 5/6, 4/6, ..., 1/6
        end
        M[101,101] = 1 # once we get to the last space, we stay there
        M
```

```
Out[2]: 101×101 Array{Float64,2}:
 0.0       0.0       0.0       0.0       …  0.0       0.0       0.0  0.0
 0.166667  0.0       0.0       0.0          0.0       0.0       0.0  0.0
 0.166667  0.166667  0.0       0.0          0.0       0.0       0.0  0.0
 0.166667  0.166667  0.166667  0.0          0.0       0.0       0.0  0.0
 0.166667  0.166667  0.166667  0.166667     0.0       0.0       0.0  0.0
 0.166667  0.166667  0.166667  0.166667  …  0.0       0.0       0.0  0.0
 0.166667  0.166667  0.166667  0.166667     0.0       0.0       0.0  0.0
 0.0       0.166667  0.166667  0.166667     0.0       0.0       0.0  0.0
 0.0       0.0       0.166667  0.166667     0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.166667     0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0       …  0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0          0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0          0.0       0.0       0.0  0.0
 ⋮                                       ⋱                           ⋮
 0.0       0.0       0.0       0.0          0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0       …  0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0          0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0          0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0          0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0          0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0       …  0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0          0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0          0.0       0.0       0.0  0.0
 0.0       0.0       0.0       0.0          0.166667  0.0       0.0  0.0
 0.0       0.0       0.0       0.0          0.166667  0.166667  0.0  0.0
 0.0       0.0       0.0       0.0       …  0.666667  0.833333  1.0  1.0
```

Now, we start on position 0, corresponding to $j = 1$. This is described by the vector

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

After one move, our probability of being on each spot is given by

$$Me_1 = \begin{pmatrix} 0 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(the first column of M).

```
In [3]: e₁ = zeros(101); e₁[1] = 1
        M*e₁
```

```
Out[3]: 101-element Array{Float64,1}:
         0.0
         0.166667
         0.166667
         0.166667
         0.166667
         0.166667
         0.166667
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         ⋮
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
```

That is, there is a 1/6 chance of being in positions 1, 2, 3, 4, 5, or 6.
After *two* moves, the probability distribution is given by $M^2 e_1$:

```
In [4]: M^2 * e₁
```

```
Out[4]: 101-element Array{Float64,1}:
         0.0
         0.0
         0.0277778
         0.0555556
         0.0833333
         0.111111
         0.138889
         0.166667
         0.138889
         0.111111
         0.0833333
         0.0555556
         0.0277778
         ⋮
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
```

```
0.0
0.0
0.0
0.0
0.0
0.0
```

And so on.

In fact, the matrix M is precisely a **Markov matrix.** It has the property that the **sum of every column is 1**, as can be checked in Julia by:

```
In [5]: sum(M, 1) # sum M along the first dimension, i.e. sum M[U+1D62][U+2C7C] over i, i.e. sum each c
```

```
Out[5]: 1×101 Array{Float64,2}:
         1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  …  1.0  1.0  1.0  1.0  1.0  1.0  1.0
```

The eigenvalues of this matrix are weird looking: there is a single steady state (eigenvalue 1), and all other eigenvalues are zero!

```
In [6]: eigvals(M)
```

```
Out[6]: 101-element Array{Float64,1}:
         1.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         ⋮
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
```

What is actually happening here is that this matrix is not diagonalizable — it is defective. The matrix X of eigenvectors is singular:

```
In [7]: λ, X = eig(M)
        det(X)
```

```
Out[7]: 0.0
```

Let's not worry about that for now (we will cover defective matrices later), and instead focus on the **steady-state eigenvalue λ=1**. The corresponding eigenvector is just the unit vector e_{101}, because the steady state is the situation where we have reached the last spot on the board, at which point we stay there forever:

```
In [8]: X[:,1]

Out[8]: 101-element Array{Float64,1}:
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         ⋮
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         0.0
         1.0
```

Let's plot this probability distribution as it evolves over many moves, plotting it on a 2d grid that resembles the usual Chutes and Ladders board.

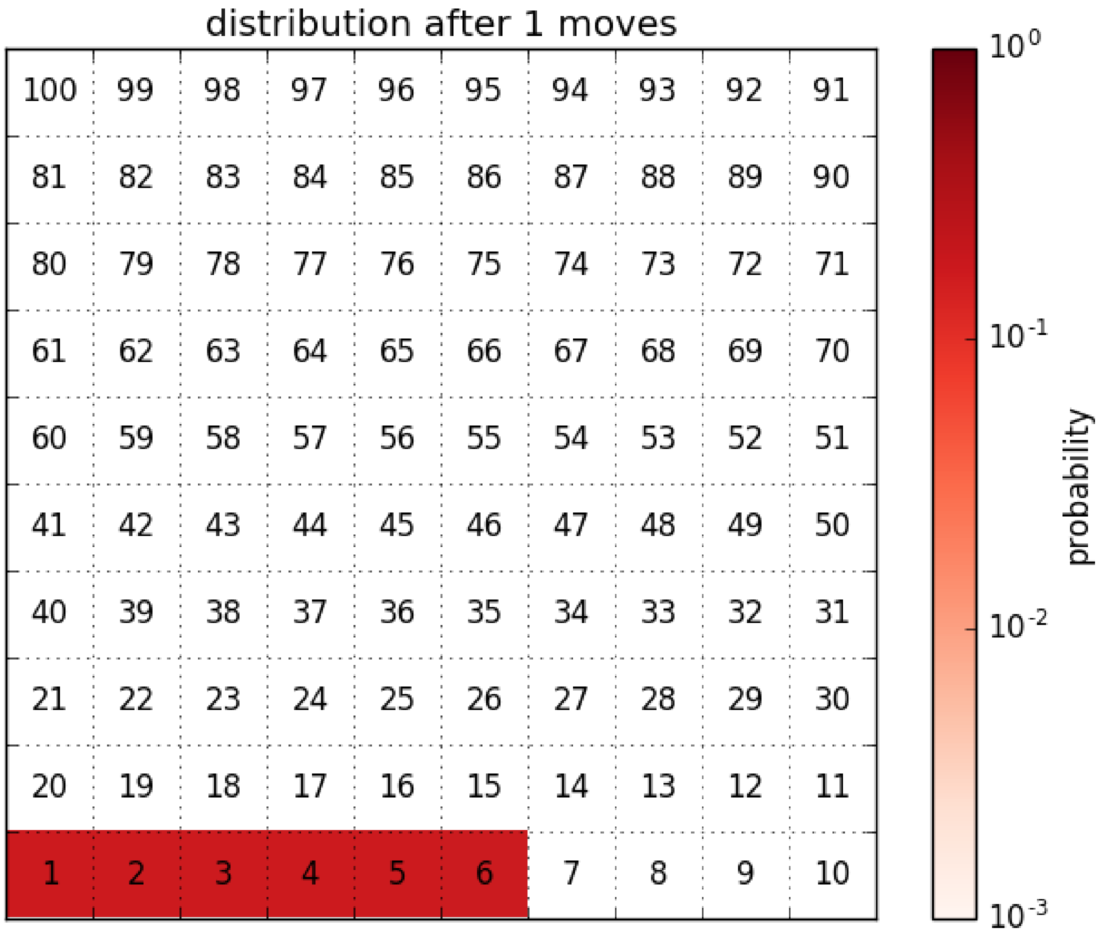
```
In [9]: # Plot the probabilities on something like a chutes and ladders board.  We won't show the start
        # since that is not on the board.
        function plotchutes(p)
            P = reshape(p[2:101], 10,10).' # reshape to a 10×10 array and transpose to row-major
            # every other row should be reversed, corresponding to how players "zig-zag" across the boa
            for i = 2:2:10
                P[i,:] = reverse(P[i,:])
            end
            imshow(P, aspect="equal", cmap="Reds", norm=PyPlot.matplotlib["colors"]["LogNorm"](vmin=1e-
            colorbar(label="probability")
            xticks(-0.5:1:10, visible=false)
            yticks(-0.5:1:10, visible=false)
            grid()
            for i = 1:10, j = 1:10
                n = (i-1)*10 + j
                x = iseven(i) ? 10-j : j-1
                y = i-1
                text(x,y, "$n", horizontalalignment="center", verticalalignment="center")
            end
        end
```

```
Out[9]: plotchutes (generic function with 1 method)
```

```
In [10]: fig = figure()
         @manipulate for n in slider(1:100, value=1)
             withfig(fig) do
                 plotchutes(M^n*e₁)
                 title("distribution after $n moves")
             end
         end
```

```
Interact.Slider{Int64}(Signal{Int64}(1, nactions=1),"",1,1:100,"horizontal",true,"d",true)
```

```
Out[10]:
```

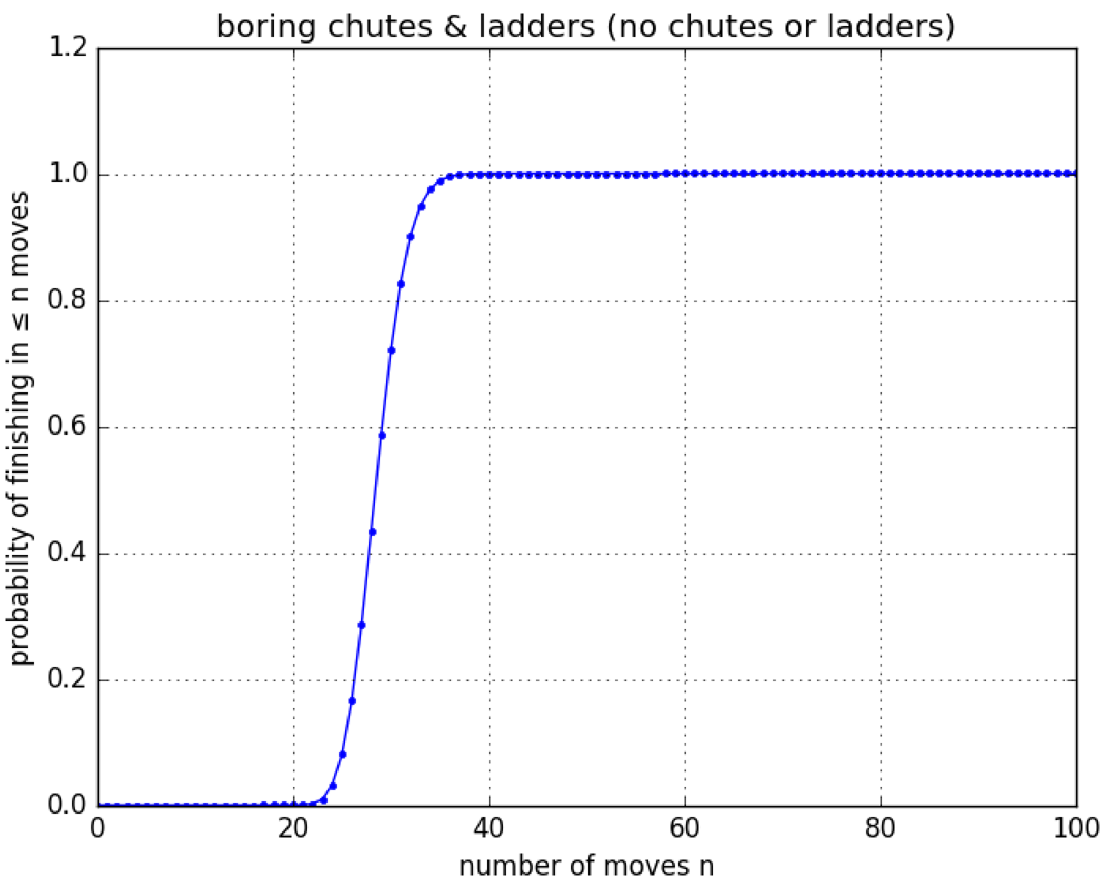


This is a boring game: you move forward monotonically along the board until you reach the end. After 100 moves, the probability of having reached the end is 100%, because on each turn you move at least 1 space forward:

```
In [11]: M^100*e₁

Out[11]: 101-element Array{Float64,1}:
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          ⋮
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          0.0
          1.0
```

We can plot the probability $e_{101}^T M^n e_1$ of finishing the game after $\leq n$ steps (with a single player) as a function of n:

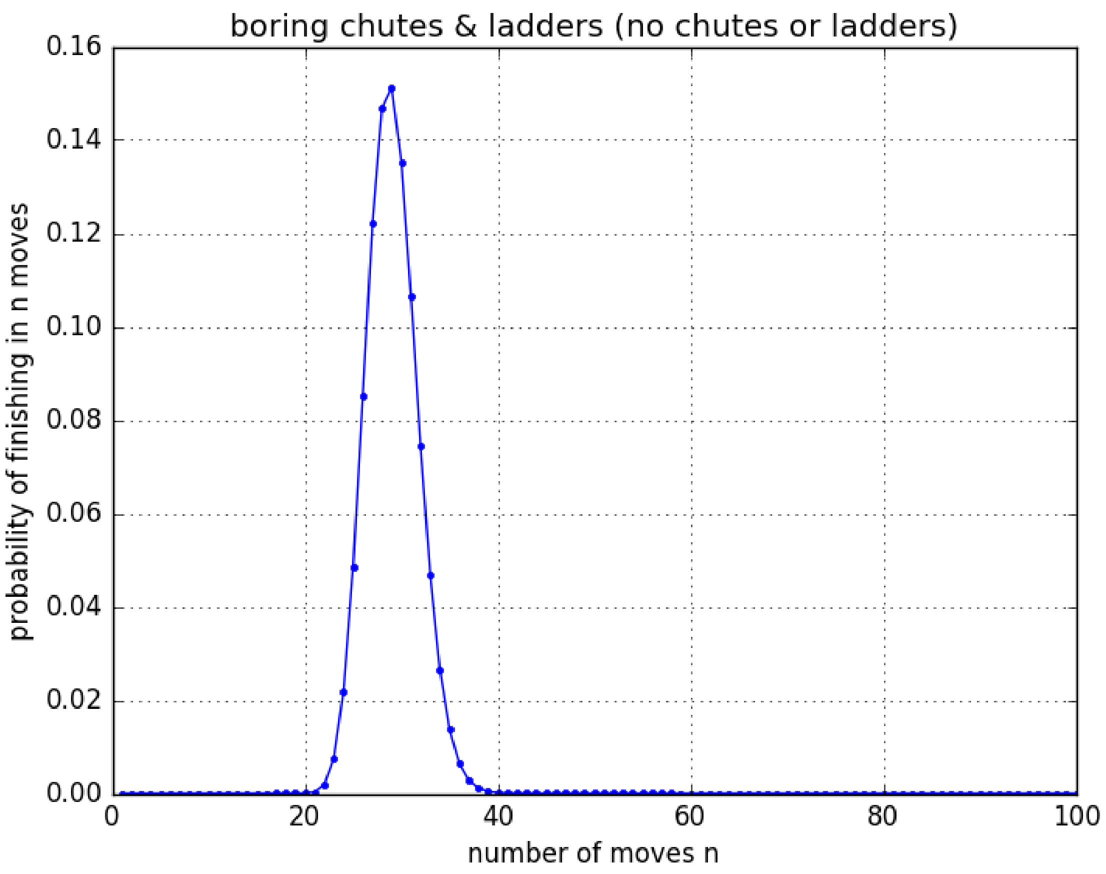
```
In [12]: plot(0:100, [(M^n * e₁)[end] for n = 0:100], "b.-")
         xlabel("number of moves n")
         ylabel("probability of finishing in ≤ n moves")
         grid()
         title("boring chutes & ladders (no chutes or ladders)")
```



```
Out[12]: PyObject <matplotlib.text.Text object at 0x325e78d50>
```

If $p(n) = e_{101}^T M^n e_1$ is the probability of finishing in $\leq n$ moves, then the probability of finishing in exactly n moves is $p(n) - p(n-1)$. The Julia `diff` function will compute this difference for us given a vector of p values:

```
In [13]: plot(1:100, diff([(M^n * e1)[end] for n = 0:100]), "b.-")
         xlabel("number of moves n")
         ylabel("probability of finishing in n moves")
         grid()
         title("boring chutes & ladders (no chutes or ladders)")
```



```
Out[13]: PyObject <matplotlib.text.Text object at 0x328aa0610>
```

And the expected number of moves is

$$\sum_{n=1}^{\infty} n[p(n) - p(n-1)]$$

```
In [14]: sum((1:100) .* diff([(M^n * e1)[end] for n = 0:100]))
```

```
Out[14]: 29.04761904761906
```

1.2 Adding chutes and ladders

Now, we will add in the effect of chutes and ladders. After you make each move represented by M above, then we additionally go up a ladder or down a chute if we landed on one. We represent this by a transition matrix T, where $T_{ij} = 1$ if there is a ladder/chute from j to i. For positions j with no chute or ladder, we set $T_{jj} = 1$.

The following is the list of chutes and ladders from the game board shown at the top:

```
In [15]: T = zeros(101,101)

         for t in (1=>39, 4=>14, 9=>31, 28=>84, 36=>44, 51=>67, 80=>100, 71=>91, # ladders
                   16=>6, 47=>26, 49=>11, 56=>53, 64=>60, 92=>73, 95=>75, 98=>78) # chutes
```

```
            T[t[2]+1,t[1]+1] = 1
        end

        # Set T[j,j] = 1 in spaces with no chute/ladder
        for j = 1:101
            if all(T[:,j] .== 0)
                T[j,j] = 1
            end
        end
```

The matrix T is also a Markov matrix!

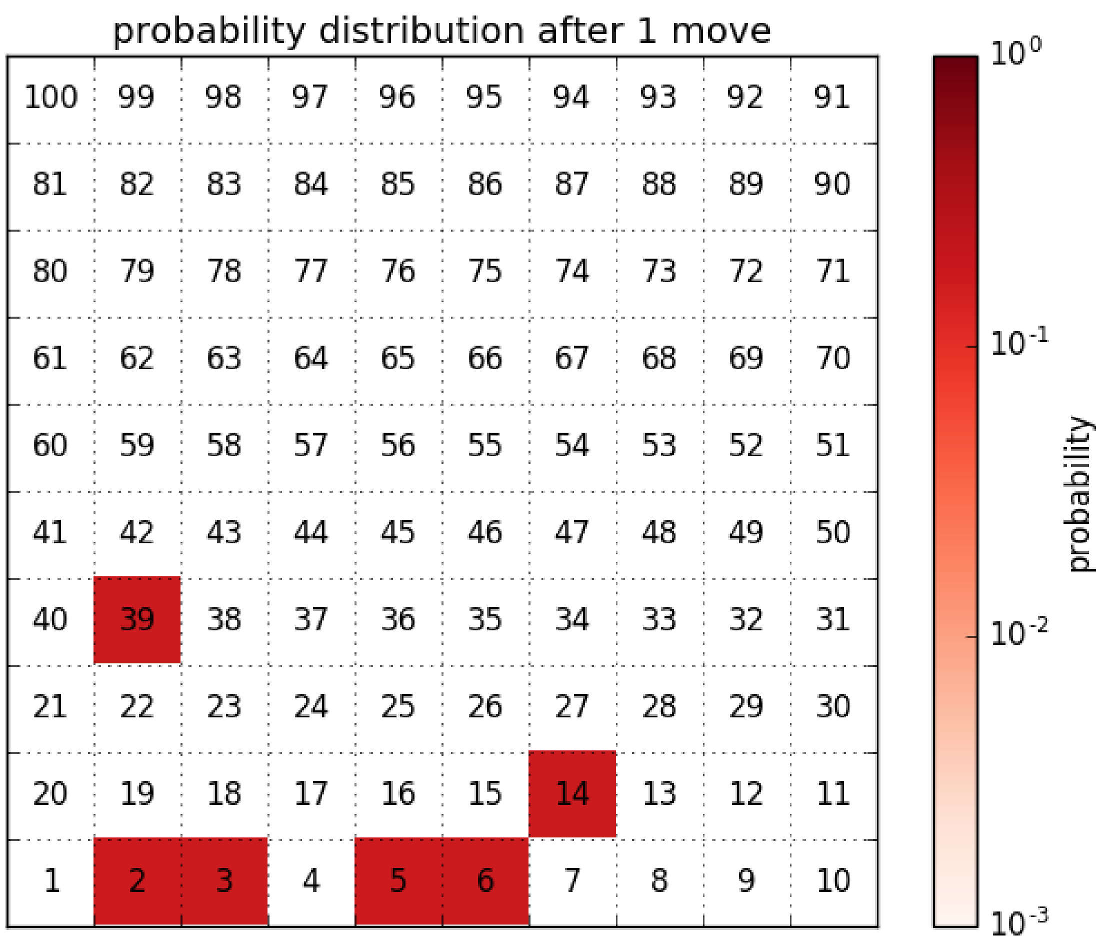
```
In [16]: sum(T,1)

Out[16]: 1×101 Array{Float64,2}:
          1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  …  1.0  1.0  1.0  1.0  1.0  1.0  1.0
```

Making the move M followed by the transition T is represented by their product TM, which is also a Markov matrix. (The product of any Markov matrices is also a Markov matrix.)

```
In [17]: sum(T*M, 1)

Out[17]: 1×101 Array{Float64,2}:
          1.0  1.0  1.0  1.0  1.0  1.0  1.0  1.0  …  1.0  1.0  1.0  1.0  1.0  1.0  1.0
```

After a single move, the probability distribution is TMe_1, and we see the effect of the two ladders that can be reached in a single move:

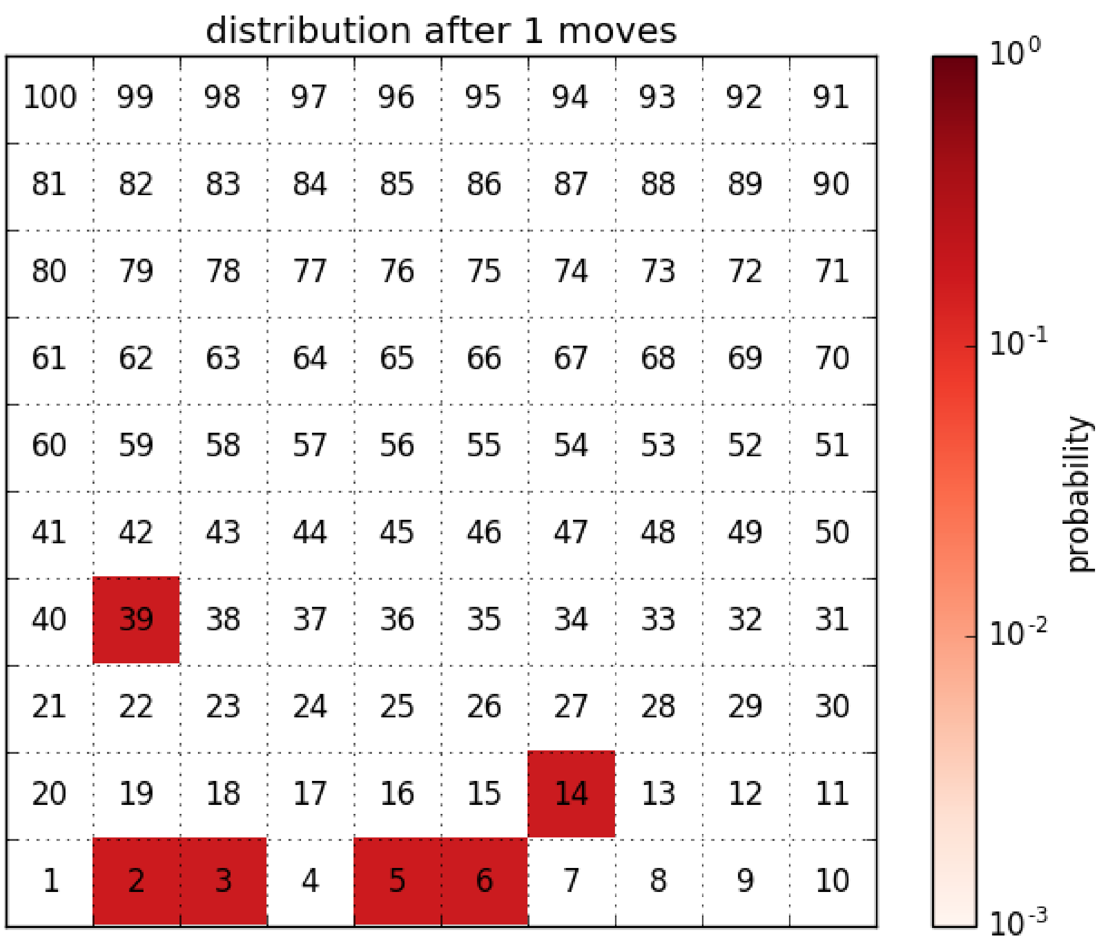
```
In [18]: plotchutes(T*M*e₁)
         title("probability distribution after 1 move")
```



```
Out[18]: PyObject <matplotlib.text.Text object at 0x329030250>
```

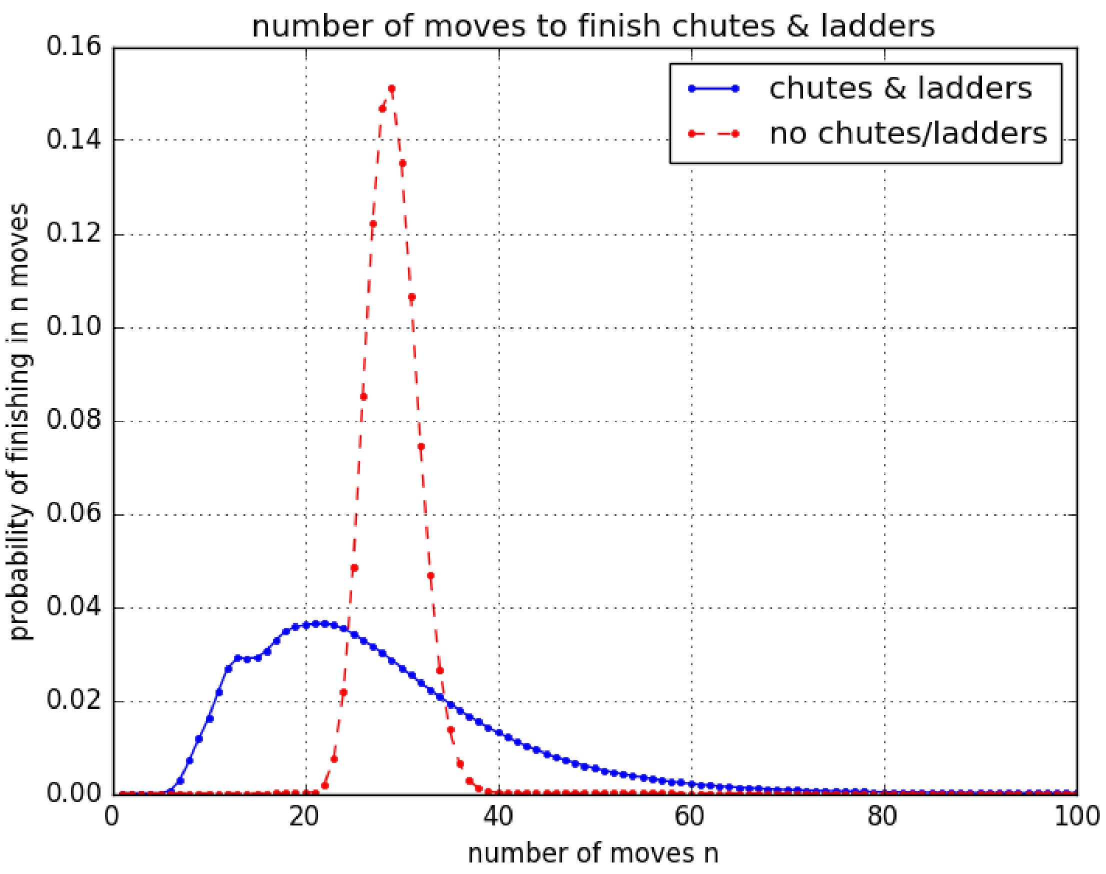
As above, the probability distribution after n moves is $(TM)^n e_1$, and it is interesting to plot this:

```
In [19]: fig = figure()
         @manipulate for n in slider(1:100, value=1)
             withfig(fig) do
                 plotchutes((T*M)^n*e₁)
                 title("distribution after $n moves")
             end
         end

Interact.Slider{Int64}(Signal{Int64}(1, nactions=1),"",1,1:100,"horizontal",true,"d",true)


  Out[19]:
```



Games can end much more quickly because of the ladders, but they can also take much longer because of the chutes. Let's plot the probability distribution vs. n as before:

```
In [20]: plot(1:100, diff([((T*M)^n * e₁)[end] for n = 0:100]), "b.-")
         plot(1:100, diff([(M^n * e₁)[end] for n = 0:100]), "r.--")
         xlabel("number of moves n")
         ylabel("probability of finishing in n moves")
         grid()
         title("number of moves to finish chutes & ladders")
         legend(["chutes & ladders", "no chutes/ladders"])
```



```
Out[20]: PyObject <matplotlib.legend.Legend object at 0x329705110>
```

The expected number of moves (for a single player) is:

```
In [21]: sum((1:1000) .* diff([((T*M)^n * e1)[end] for n = 0:1000]))
```

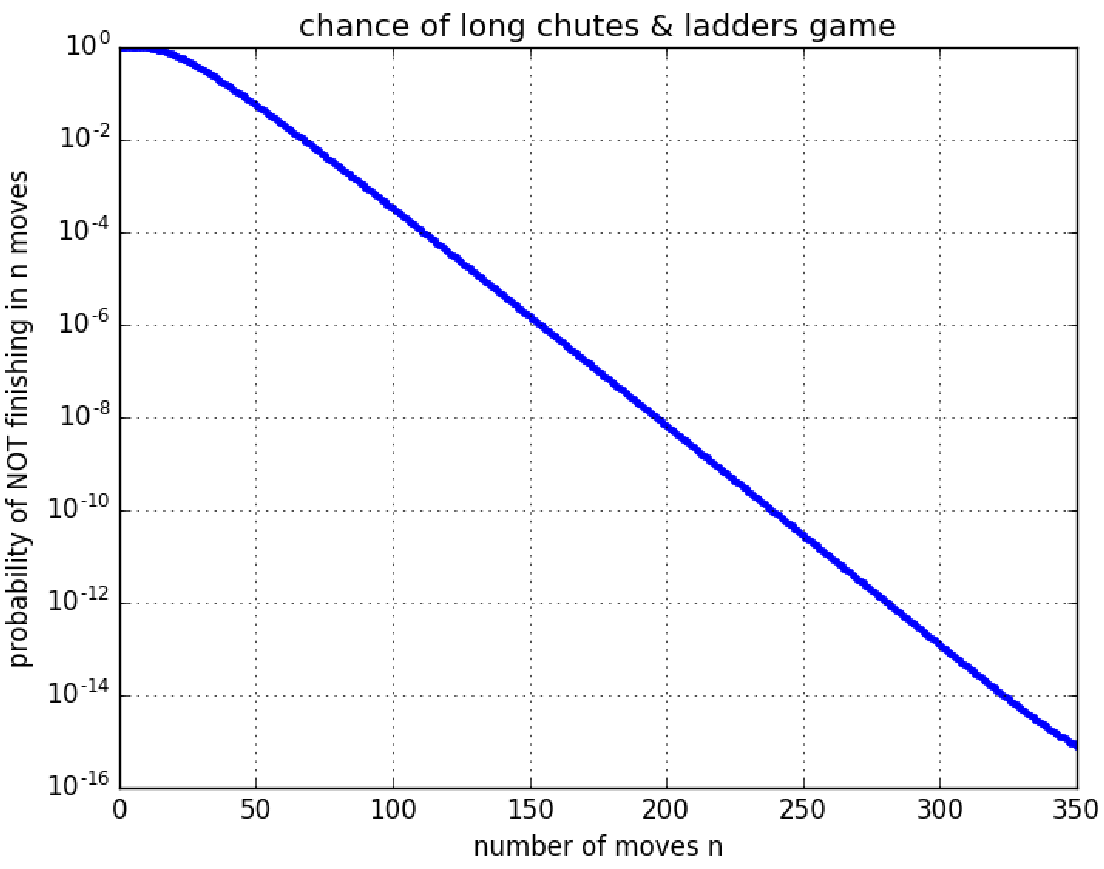
```
Out[21]: 27.130202016993316
```

Amazingly, this is about the same as the 29 moves expected when there are no chutes and ladders, but the variance is much larger!

(In principle, we should actually sum for `n=0` to ∞, but because the probability $p(n) - p(n-1)$ decays exponentially for large `n` we can just truncate the sum.)

And unlike the boring version, the probability of the game finishing never reaches 100%. If you are unlucky, you could be trapped playing chutes and ladders for all eternity! Let's plot $1 - p(n)$ vs. n:

```
In [22]: semilogy(0:350, [1-((T*M)^n * e1)[end] for n = 0:350], "b.-")
         xlabel("number of moves n")
         ylabel("probability of NOT finishing in n moves")
         grid()
         title("chance of long chutes & ladders game")
```



```
Out[22]: PyObject <matplotlib.text.Text object at 0x329a6fcd0>
```

Fortunately, the probability of a long game decreases exponentially fast with n.

2 Absorbing Markov matrix

It turns out that the matrix M (and TM) for this problem is something called an absorbing Markov matrix.

It is "absorbing" because the final position 101 (spot 100 on the board) cannot be escaped, and can be reached from every other position. This has two consequences:

- Every initial vector eventually reaches this "absorbing" steady state, even though it is not a positive Markov matrix.
- There are nice analytical formulas for the expected number of moves, the variance, etcetera. We don't actually have to sum up $n[p(n) - p(n-1)]$ as above.

Deriving these nice formulas is not too hard, but is a bit outside the scope of 18.06. But, just for fun, here is the "clever way" to compute the expected number of moves to finish *Chutes & Ladders*:

```
In [23]: A = (T*M)'[1:100,1:100]  # the 100x100 upper-left corner of (TM)[U+1D40]
         N = inv(I - A)           # N[i,j] = expected number of visits to i starting at j
         (N * ones(100))[1]       # expected number of moves to finish starting at 1
```

```
Out[23]: 27.130202016993298
```

This matches our brute-force calculation from above!