Projections

September 7, 2017

```
In [1]: using PyPlot, Interact
```

0.1 Projection onto a line

Suppose b is a vector of data and we want to find p, a multiple of $a = (1, 1, \ldots, 1)$, say, closest to b. Which vector is that?

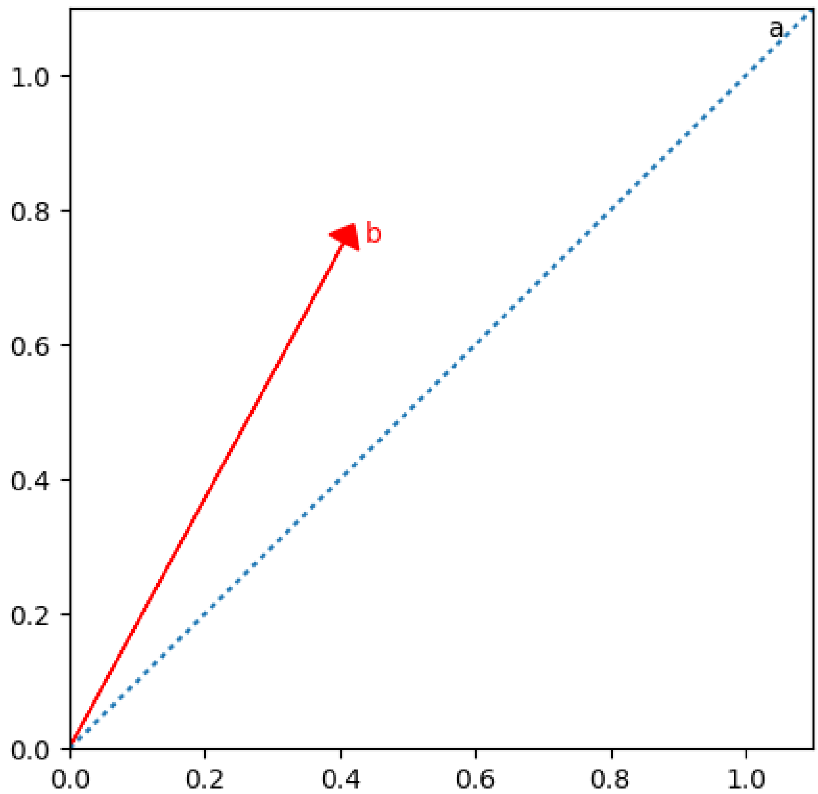
Let us call this vector $p = \hat{x}a$.

Here is an example in 2d:

```
In [2]: b = rand(2)
```

```
Out[2]: 2-element Array{Float64,1}:
         0.405977
         0.752382
```

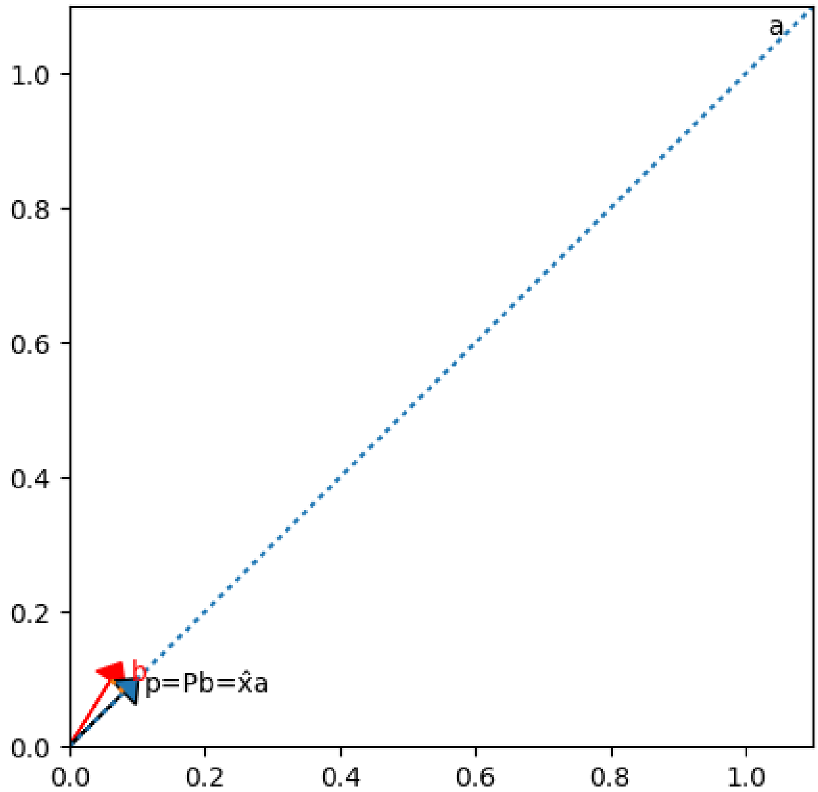
```
In [3]: figure(figsize=(5,5))
        arrow(0,0,b[1],b[2],head_width=0.05, head_length=0.03,color="r")
        plot([0,1.1],[0,1.1],":")
        text(b[1]+.03,b[2],"b",color="r")
        text(1.03,1.06,"a")
        axis([0,1.1,0,1.1]);
```




```
In [4]: b = rand(2)       # random red vector


        figure(figsize=(5,5))
        arrow(0,0,b[1],b[2],head_width=0.05, head_length=0.03,color="r")
        text(b[1]+.03,b[2],"b",color="r")
        plot([0,1.1],[0,1.1],":")
        axis([0,1.1,0,1.1]);

        a = ones(2)     # target direction
        x^ = (a'b)/(a'a)
        p = a * x^       # projection
        plot([b[1],p[1]],[b[2],p[2]],":")
        arrow(0,0,p[1],p[2],head_width=0.05, head_length=0.03)
        text(p[1]+.03,p[2],"p=Pb=x^a")
        text(1.03,1.06,"a")
```



Out[4]: `PyObject <matplotlib.text.Text object at 0x329310c50>`

Let us break this into steps

1. Find $\hat{x}$
2. Find p
3. Find matrix P such that $Pb = p$

To do this form the "error" vector $e = b - p = b - \hat{x}a$ where $\hat{x}$ is the unknown. We choose $\hat{x}$ specifically to make $e \perp a$.

We want $a \cdot (b - \hat{x}a) = 0$ (where $\cdot$ denotes the dot product) so that

1. \$ = (a $\cdot b)/(a \cdot a) = (a^{\mathrm{Tb})/(^{\mathrm{a}}}Ta)\$wethenhavep = \hat{x}a = a\hat{x} = a(a^Tb)/(a^Ta)$
3. $Pb = a(a^Tb)/(a^Ta)$ gives $P = (aa^T)/(a^Ta)$

Example:

In [5]:
```
P = (a*a')/(a'a)
```

Out[5]:
```
2×2 Array{Float64,2}:
 0.5  0.5
 0.5  0.5
```

In [6]:
```
@manipulate for n=slider(1:15,value=2)
    a = ones(Rational,n)
    P = (a*a')/(a'a)
end
```

```
Interact.Slider{Int64}(Signal{Int64}(2, nactions=1),"",2,1:15,"horizontal",true,"d",true)
```

```
Out[6]: 2×2 Array{Rational{Int64},2}:
         1//2  1//2
         1//2  1//2
```

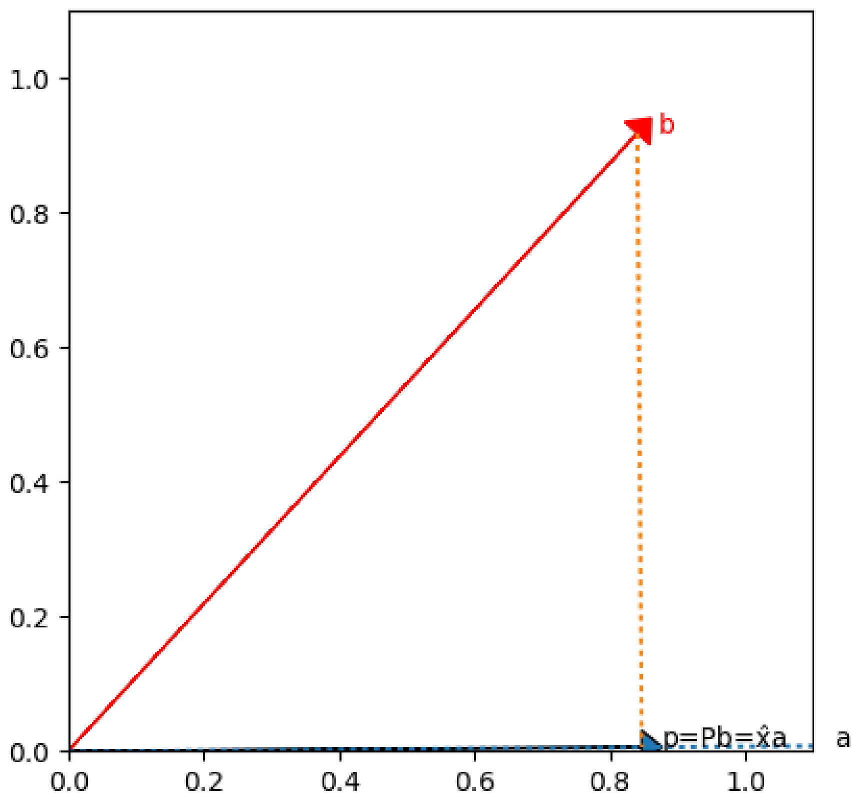
In the special case of a being the ones vector, $\hat{x}$ is the mean of b. If only one number is used to summarize a large data vector b, it is commonly the mean.

Now consider more general a.

```
In [7]: b = rand(2)                        # random red vector
        a = rand(2); a *= 1.1/maximum(a)   # target direction
        P = (a*a')/a'a
        p = P*b

        figure(figsize=(5,5))
        arrow(0,0,b[1],b[2],head_width=0.05, head_length=0.03,color="r")
        text(b[1]+.03,b[2],"b",color="r")
        plot([0,a[1]],[0,a[2]],":")
        axis([0,1.1,0,1.1]);

        plot([b[1],p[1]],[b[2],p[2]],":")
        arrow(0,0,p[1],p[2],head_width=0.05, head_length=0.03)
        text(p[1]+.03,p[2],"p=Pb=x^a")
        text(a[1]+.03,a[2],"a")
```



Out[7]: PyObject <matplotlib.text.Text object at 0x3296e27d0>

```
In [8]: # Powers of P remain equal.  Explain why geometrically?
        # Answer: once you project, projecting again keeps you where you were
        display(P)
        display(P^2)
        display(P^3)
```

```
2×2 Array{Float64,2}:
 0.999947    0.00729359
 0.00729359  5.31993e-5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 0.999947    0.00729359
 0.00729359  5.31993e-5


2×2 Array{Float64,2}:
 0.999947    0.00729359
 0.00729359  5.31993e-5
```

Relationship to least squares:

```
In [9]: x^= (a'b)/(a'a)
```

```
Out[9]: 1×1 Array{Float64,2}:
         0.770082
```

```
In [10]: a\b
```

```
Out[10]: 1-element Array{Float64,1}:
          0.770082
```

0.2 Projection on a subspace

```
In [11]: A = rand(5, 3) # consider the subspace spanned by the columns of A
```

```
Out[11]: 5×3 Array{Float64,2}:
          0.857652  0.517855   0.280477
          0.418829  0.556069   0.964192
          0.376845  0.64954    0.692936
          0.839149  0.0958249  0.297049
          0.533046  0.988303   0.900709
```

```
In [12]: b = rand(5)
```

```
Out[12]: 5-element Array{Float64,1}:
          0.65487
          0.536363
          0.847276
          0.527487
          0.667052
```

Our problem 1. Find the vector p that is in the column space of A that is closest to b 2. Project b onto the column space of A

Find the linear combination of the columns of $(m \times n)$ A closest to b

In other words, find an $\hat{x}$ in $\Re^n$ such that $A\hat{x}$ is closest to b.

How do we find $\hat{x}$? Idea is the same as the line. Make \$e=b-A $\perp$ \$$toeverycolumnofA$:

$A^T(b - A\hat{x}) = 0$ is equivalent to the first column of A is orthogonal to e, and the second column is orthogonal to e, ..., and the last column of A is orthogonal to A.

$A^T A\hat{x} = A^T b$. (known as the **normal equations**)

1. $\hat{x} = (A^T A)^{-1} A^T b$
2. $p = A\hat{x} = A(A^T A)^{-1} A^T b$
3. \$P =A(A^{TA)}{-1}A^T \$ (is the projection matrix)

Some examples

```
In [13]: A
```

```
Out[13]: 5×3 Array{Float64,2}:
 0.857652  0.517855   0.280477
 0.418829  0.556069   0.964192
 0.376845  0.64954    0.692936
 0.839149  0.0958249  0.297049
 0.533046  0.988303   0.900709
```

```
In [14]: P = A * inv(A'A) * A'
```

```
Out[14]: 5×5 Array{Float64,2}:
  0.680624   -0.275029   0.0718287   0.273674    0.24835
 -0.275029    0.726659   0.227343    0.235613    0.125645
  0.0718287   0.227343   0.233565   -0.0612794   0.344112
  0.273674    0.235613  -0.0612794   0.765488   -0.212956
  0.24835     0.125645   0.344112   -0.212956    0.593663
```

```
In [15]: P^10
```

```
Out[15]: 5×5 Array{Float64,2}:
  0.680624   -0.275029   0.0718287   0.273674    0.24835
 -0.275029    0.726659   0.227343    0.235613    0.125645
  0.0718287   0.227343   0.233565   -0.0612794   0.344112
  0.273674    0.235613  -0.0612794   0.765488   -0.212956
  0.24835     0.125645   0.344112   -0.212956    0.593663
```

```
In [16]: b = rand(5)
```

```
Out[16]: 5-element Array{Float64,1}:
 0.458361
 0.601126
 0.721415
 0.940134
 0.816022
```

```
In [17]: p = P*b
```

```
Out[17]: 5-element Array{Float64,1}:
 0.658412
 0.798797
 0.561274
 0.768752
 0.721845
```

```
In [18]: e = p - b

Out[18]: 5-element Array{Float64,1}:
          0.200051
          0.197671
         -0.16014
         -0.171382
         -0.094177

In [19]: A'e

Out[19]: 3-element Array{Float64,1}:
         -1.49186e-15
         -1.22125e-15
         -1.67921e-15

In [20]: x^ =  inv(A'A)*A'b

Out[20]: 3-element Array{Float64,1}:
          0.707741
         -0.265997
          0.674437

In [21]: A\b # in matlab and in julia, to solve the least squares system
         # Ax=b for the best vector x^, type A\b

Out[21]: 3-element Array{Float64,1}:
          0.707741
         -0.265997
          0.674437
```

0.3 Math: (A^TA) is invertible when A has linearly independent columns

Suppose that A^TA is not invertible. Then there is a nonzero x x such that $A^TAx = 0$. Then $x^TA^TAx = 0 = \|Ax\|^2$. Then $Ax = 0$ meaning A does not have linearly independent columns. Taking the contrapositive, if A has linearly independent columns A^TA is invertible.

Note logically one should prove the converse too. This is implied in the "when." If A does not have linearly independent columns, there is a nonzero x with $Ax = 0$. Multiplying by A^T we have A^TAx is then 0 so A^TA is not invertible.

0.4 Briefly mentioned:

- Chebychev Approximation = polynomial fitting = linear equations
- Machine learning = nonlinear fitting = nonlinear equations
- In high school stats classes , students are told to divide by $n - 1$, not n, for sample variance.
- Some argument about degrees of freedom usually appeases the masses. In fact, the projection matrix `P = I - ones(n,n)/n` can be viewed as "removing the mean" or projection orthogonal to the "ones" vector. Removing the true mean creates a vector whose element squares have expectation σ^2 and cross terms have expectation 0.
- You might check that the sample variance numerator is $\|Pb\|^2$. This is the same as b^TPb, which is readliy checked to have average σ^2 times the sum of the diagonal elements of P, which is $n \times \left(1 - \frac{1}{n}\right) = n - 1$.