MIT 18.06 Exam 1, Spring 2017

Your name: ______________________________

Recitation: ______________________________

problem	**score**
1	/25
2	/25
3	/25
4	/25
total	/100

Problem 1:

Suppose A is the 6×6 matrix

$$A = \begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix}.$$

(a) What is the rank of A? (*Hint:* doing elimination is okay. You should notice a simple pattern.)

(b) Give a basis for $N(A)$.

(c) For what $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix}$ does $Ax = b$ have a solution? Give an equation in terms of the entries $b_1, \ldots, b_6$.

Hint: from class, if we do the same row operations to transform $b \rightsquigarrow c$ as we did for Gaussian elimination to transform $A \rightsquigarrow U$ (or R), for $b \in C(A)$ we needed c to be ______________ in the rows where U is ______________. For *this* A, the row operations have a simple pattern you should have noticed above.

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Problem 2:

Circle which of the following statements *might possibly* be true. *Give an example* of a possible matrix A *for each possibly true* statement.

(a) $Ax = b$ has a unique solution for a 5×3 matrix A.

(b) $Ax = b$ has a unique solution for a 3×5 matrix A.

(c) $Ax = b$ is not solvable for *any* b.

(d) $Ax = b$ is not solvable for any $b \neq 0$.

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Problem 3:

Suppose that we do *column operations* on the matrix A to transform it to another matrix B:

$$\underbrace{\begin{pmatrix} 2 & 4 & 6 \\ 3 & 1 & 10 \\ 0 & -1 & 3 \end{pmatrix}}_{A} \rightsquigarrow \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 3 & -5 & 1 \\ 0 & -1 & 3 \end{pmatrix}}_{B}.$$

For example, we subtracted twice the first column of A from the second column of A to get the second column of B.

(a) Write B as a matrix product involving A and some other matrix.

(b) Which of $C(A)$ and $N(A)$ are the same as $C(B)$ and $N(B)$, if any? (No computation should be required! *You don't have to compute these subspaces explicitly*!)

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Problem 4:

Suppose you are *given* the $PA = LU$ factorization of an invertible $n \times n$ matrix A. Now, suppose we want to solve

$$\begin{pmatrix} A & B \\ 0 & A \end{pmatrix} x = b$$

for some $n \times n$ matrix B, where "0" denotes an $n \times n$ block of zeros in the lower-left corner.

(a) Suppose we express $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where x_1 and x_2 are n-component *vectors*. Similarly, we express $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ in terms of n-component vectors b_1 and b_2.

Write the solution x_1 and x_2 in terms of P, L, U, B, b_1, b_2 (or the inverses of those matrices). *Hint:* write out two $n \times n$ equations involving x_1 and x_2 first.

(b) Take your answer from (a) and explain how (if you do things *in the right order*), you can compute the solution x in $\sim n^2$ operations (i.e. roughly proportional to n^2).

You can indicate the order of operations by parentheses. For example, if you have an expression LBb_1 in your answer, you could either evaluate it as $(LB)b_1$ (multiply LB, then multiply by b_1) or evaluate it as $L(Bb_1)$ (multiply Bb_1, then multiply L by that vector).

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