Problem 1:

The following matrix $M$ is a Markov matrix (its columns sum to 1):

$$M = \begin{pmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{pmatrix}$$

and its steady-state eigenvector ($\lambda = 1$) is

$$s = \begin{pmatrix} 7/18 \\ 1/3 \\ 5/18 \end{pmatrix}.$$  

Recall from class that multiplying a vector $x$ by $M$ does not change the sum of the components. That is, the sum $= o^T x$, where $o = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, is conserved:

$$o^T M x = o^T x = x_1 + x_2 + x_3$$

for any $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. (The steady-state vector $s$ above was normalized so that $o^T s = 1$.)

(a) If we let $x = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, what vector does $M^n x$ approach as $n \to \infty$?

(b) For the same $x$, in what direction does $(M^T)^n x$ point as $n \to \infty$. (You don’t have to give the correct magnitude, just give a vector in the correct direction.)

(c) Multiplying $M^T x$ does not conserve the sum of the components of $x$, unlike $M x$. However, it does conserve some linear combination of the components: there is some vector $v \neq 0$ such that

$$v^T M^T x = v^T x$$

for all $x$. What is $v$? (Hint: this is easy if you understand why $o^T M x = o^T x$ as stated above.)
Problem 2:

Suppose that $A$ is a $3 \times 3$ real-symmetric matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -2$, and corresponding eigenvectors $x_1, x_2, x_3$. You are given that $x_1 = (1, 0, 1)$.

(a) Give an approximate solution at $t = 100$ to $\frac{dx}{dt} = Ax$ for $x(0) = (1, 1, 0)$. (You should give a specific vector, even if the vector is very big or very small — an answer of “$\approx 0$” or “$\approx \infty$” is not acceptable.)

(b) If $x_2 = (0, 1, 0)$, what is $x_3$? (You should not need your answer here to solve the previous part!)

(c) If instead we solve $\frac{dx}{dt} = (\alpha I - A^3)x$ for some complex number $\alpha$ and the same $x(0)$, give a possible value of $\alpha$ for which the solutions $x(t)$ approach oscillating solutions (but not decaying or growing!) for large $t$. 


(blank page for your work if you need it)
Problem 3:

The real $3 \times 3$ matrix $A$ is positive-definite, and the real $3 \times 4$ matrix $B$ is rank 3:

\[ B = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & -1 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \]

The nullspace $N(B)$ is spanned by the vector $x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

(a) How many zero, positive, and negative eigenvalues does $B^T AB$ have? (Hint: what happens if you plug an eigenvector into $x^T (B^T AB)x$?)

(b) For which sign (+ or −) does $\frac{dx}{dt} = \pm B^T ABx$ have solutions that approach a constant steady state for any initial condition $x(0)$?

(c) For the sign you chose in the previous part, what is $x(\infty)$ for $x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$?
(blank page for your work if you need it)
Problem 4:

True or false. Give a reason if true (one sentence and/or one equation should suffice), or a counterexample if false.

(a) A singular matrix $A$ cannot be similar to a non-singular matrix $B$.

(b) Any positive markov matrix $M$ (that is, positive entries) must also be positive definite.

(c) If $A = QR$ is the QR factorization of a real (square) matrix $A$, then the matrix $RQ$ has the same eigenvalues as $A$.

(d) $A$ and $e^{A^3}$ have the same eigenvalues.

(e) $A$ and $e^{A^3}$ have the same eigenvectors.