MIT 18.06 Exam 3, Spring 2017

Your name: ______________________________

Recitation: ______________________________

problem	**score**
1	/25
2	/25
3	/25
4	/25
total	/100

Problem 1:

The following matrix M is a Markov matrix (its columns sum to 1):

$$M = \begin{pmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{pmatrix}$$

and its steady-state eigenvector ($\lambda = 1$) is

$$s = \begin{pmatrix} 7/18 \\ 1/3 \\ 5/18 \end{pmatrix}.$$

Recall from class that multiplying a vector x by M does not change the sum of the components. That is, the sum $= o^T x$, where $o = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, is conserved:

$$o^T M x = o^T x = x_1 + x_2 + x_3$$

for any $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. (The steady-state vector s above was normalized so that $o^T s = 1$.)

(a) If we let $x = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, what vector does $M^n x$ approach as $n \to \infty$?

(b) For the same x, in what direction does $(M^T)^n x$ point as $n \to \infty$. (You don't have to give the correct *magnitude*, just give a vector in the correct *direction*.)

(c) Multiplying $M^T x$ does *not* conserve the *sum* of the components of x, unlike Mx. However, it does conserve *some* linear combination of the components: there is some vector $v \neq 0$ such that

$$v^T M^T x = v^T x$$

for all x. What is v? (Hint: this is easy if you understand *why* $o^T M x = o^T x$ as stated above.)

(blank page for your work if you need it)

Problem 2:

Suppose that A is a 3×3 real-symmetric matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -2$, and corresponding eigenvectors x_1, x_2, x_3. You are given that $x_1 = (1, 0, 1)$.

(a) Give an approximate solution at $t = 100$ to $\frac{dx}{dt} = Ax$ for $x(0) = (1, 1, 0)$. (You should give a specific vector, even if the vector is very big or very small — an answer of "≈ 0" or "$\approx \infty$" is not acceptable.)

(b) If $x_2 = (0, 1, 0)$, what is x_3? (You should *not* need your answer here to solve the previous part!)

(c) If instead we solve $\frac{dx}{dt} = (\alpha I - A^3)x$ for some *complex* number $\alpha$and the same $x(0)$, give a possible value of α for which the solutions $x(t)$ approach *oscillating* solutions (but not decaying or growing!) for large t.

(blank page for your work if you need it)

Problem 3:

The real 3×3 matrix A is positive-definite, and the real 3×4 matrix B is rank 3:

$$B = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 2 & -1 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The nullspace $N(B)$ is spanned by the vector $x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

(a) How many zero, positive, and negative eigenvalues does $B^T AB$ have? (Hint: what happens if you plug an eigenvector into $x^T(B^T AB)x$?)

(b) For which sign (+ or −) does $\frac{dx}{dt} = \pm B^T ABx$ have solutions that approach a constant steady state for any initial condition $x(0)$?

(c) For the sign you chose in the previous part, what is $x(\infty)$ for $x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$?

(blank page for your work if you need it)

Problem 4:

True or false. Give a reason if true (one sentence and/or one equation should suffice), or a counterexample if false.

(a) A singular matrix A cannot be similar to a non-singular matrix B.

(b) Any positive markov matrix M (that is, positive entries) must also be positive definite.

(c) If $A = QR$ is the QR factorization of a real (square) matrix A, then the matrix RQ has the same eigenvalues as A.

(d) A and e^{A^3} have the same eigenvalues.

(e) A and e^{A^3} have the same eigenvectors.