MIT 18.06 Final Exam, Spring 2017

Your name: ____________________

Recitation: ____________________

problem	**score**
1	/25
2	/25
3	/25
4	/25
5	/25
6	/25
7	/25
total	/175

Problem 1:

For some real matrix A, the following vectors form a basis for its column space and null space:

$$C(A) = \text{span}\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \rangle,$$

$$N(A) = \text{span}\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \rangle.$$

(a) What is the size $m \times n$ of A, what is its rank, and what are the dimensions of $C(A^T)$ and $N(A^T)$?

(b) Give *one* possible matrix A with this $C(A)$ and $N(A)$.

(c) Give a right-hand side b for which $Ax = b$ has a solution, and give *all* the solutions x for your A from the previous part. (Hint: you should not have to do Gaussian elimination.)

(d) For $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, the equation $Ax = b$ as *no solutions*. Instead, give another right-hand side $\hat{b}$ for which $A\hat{x} = \hat{b}$ is solvable and yields a least-square solution $\hat{x}$ (i.e. $\hat{x}$ minimizes $\|Ax - b\|$). $\hat{b}$ must be the ____________________________ of b onto the subspace _________________.
(Hint: if you find yourself solving a 4×4 system of equations, you are missing a way to do it much more easily. The answer should *not* depend on your choice of A matrix in part b.)

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Problem 2:

Suppose you have 100 data points (x_i, y_i) for $i = 1, 2, \ldots, 100$, and you want to fit them to a power-law curve $y(x) = ax^b$ for some a and b. Equivalently, you want to fit $\log y_i$ to $\log y = \log(ax^b) = b \log x + \log a$. Describe how to find a and b to *minimize the sum of the squares of the errors:*

$$s(a, b) = \sum_{i=1}^{100} \left(b \log x_i + \log a - \log y_i\right)^2 .$$

Write down a 2×2 system of equations for the vector $z = \begin{pmatrix} b \\ \log a \end{pmatrix}$; you can leave your equations in the form of a product of matrices/vectors as long as you say what the matrices/vectors are. (Hint: rewrite it as an 18.06-style least-squares problem with matrices/vectors.)

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Problem 3:

Suppose that 4×4 real matrix $A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix}$ has four *orthogonal* but *not orthonormal* columns a_i with lengths $\|a_1\| = 2$, $\|a_2\| = 1$, $\|a_3\| = 3$, $\|a_4\| = 2$. (That is, $a_i^T a_j = 0$ for $i \neq j$.)

(a) Write an explicit expression for the solution x to $Ax = b$ in terms of dot products, additions, and multiplications by scalars.

(b) Write A as a sum of four rank-1 matrices.

(c) If we write the matrix $B = A \begin{pmatrix} 3 & & & \\ & 6 & & \\ & & 2 & \\ & & & 3 \end{pmatrix}$, then what is $B^T B$?

Hence, for any $x \neq 0$, $\frac{\|Bx\|}{\|x\|} =$ ______.

(d) Write the SVD $A = U\Sigma V^T$: explicitly give the singular values σ (diagonal of Σ) and the singular vectors (columns of U, V, possibly in terms of the columns a_i). Hint: what is $A^T A$, and what are its eigenvectors (this should give you either U or V) and eigenvalues (related somehow to σ)? Recall also from homework that $AV = U\Sigma$.

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Problem 4:

Suppose that A and B are two real $m \times m$ matrices, and B is invertible.

(a) Circle which (if any) of the following must be true: $C(A) = C(AB)$, $C(A) = C(BA)$, $N(A) = N(AB)$, $N(A) = N(BA)$.

(b) Circle which (if any) of the following matrices must be symmetric: $ABBA$, $A^T BB^T A$, $A^T BAB^T$, $A^T BB^{-1} A$, $A^T (B^T)^{-1} B^{-1} A$?

(c) If B is a projection matrix (as well as invertible), then $AB =$____________.

(d) Do AB and BA have the same eigenvalues? Give a reason if true, a counter-example if false.

(e) Suppose A has rank r. Say as many true things as possible about the eigenvalues of $C = A^T B^T BA$ that would *not* necessarily be true if C were just a random $m \times m$ matrix.

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Problem 5:

You have a matrix A with the factorization:

$$A = \begin{pmatrix} 1 & & \\ 3 & 2 & \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ & 2 & -1 \\ & & 2 \end{pmatrix}.$$

(a) What is the product of the 3 eigenvalues of A?

(b) Solve $Ax = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$ for x. (Hint: if you find yourself doing Gaussian elimination, you are missing something.)

(c) Gaussian elimination (without row swaps) produces an $A = LU$ factorization, but you can tell at a glance that this is *not* the same as the factorization above, because L is always a lower-triangular matrix with 1's on the diagonal. Find the ordinary LU factorization of A (the matrices L and U) by multiplying and/or dividing the factors above with some diagonal matrix.

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Problem 6:

(a) One of of the eigenvalues of

$$A = \begin{pmatrix} 3 & 1 & 4 \\ & 1 & 5 \\ & 1 & 5 \end{pmatrix}.$$

is $\lambda = 3$. Find the other two eigenvalues. (Hint: one eigenvalue should be obvious from inspection of A, since A is ________________. You shouldn't need to explicitly write down and solve any cubic equation, because once you find two eigenvalues you can get the third from the ________________ of A.)

(b) For a 4×4 matrix B, the polynomial $\det(B - \lambda I)$ has three roots $\lambda = 1, 0.4, -0.7$. You find that, for some vector x, the vector $B^n x$ is getting longer and longer as n grows. It must be the case that B is a ____________________________ matrix. Approximately what is $\|B^{2000}x\|/\|B^{1000}x\|$?

(c) A positive Markov matrix M has a steady-state eigenvector $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}$. What is $M^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ for $n \to \infty$?

(d) For a real matrix C, and almost any randomly chosen initial $x(0)$, the equation $\frac{dx}{dt} = Cx$ has solutions that are *oscillating and decaying* (for large t) as a function of t. **Circle all of the things** that could **possibly be true** of C: symmetric, anti-symmetric, orthogonal, Markov, diagonalizable, defective, singular, diagonal.

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Problem 7:

The matrix A is real-symmetric and positive-definite. Using it, we write a recurrence equation

$$x_n - x_{n+1} = A\,(x_n + x_{n+1})$$

for a sequence of vectors $x_0, x_1, \ldots$

(a) For any x_n, the recurrence relation above defines a unique solution x_{n+1}. Why?

(b) If $Av = \lambda v$ and $x_0 = v$, give an equation for x_n in terms of v, n, and λ.

(c) For an arbitrary x_0, does the length of the solution $\|x_n\|$ grow with n, decay with n, oscillate, or approach a nonzero steady-state?

(d) Suppose A is 4×4 and the eigenvalues are $\lambda_1 = 0.1$, $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = 3$, and the corresponding eigenvectors are v_1, v_2, v_3, and v_4 (all normalized to length $\|v_k\| = 1$). The initial x_0 is some arbitrary vector. Write an exact formula for x_n in terms of x_0 and these eigenvectors and eigenvalues—you should get a sum of four terms. Which term should typically dominate for large n?

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