pset2-sol

September 7, 2017

1 Problem Set 2 - Solutions

1.1 Problem 1

(From Strang, section 2.2, problem 25.)

$$A = \begin{pmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{pmatrix}?$$

For which three numbers a will elimination fail to give three pivots for this matrix? That is, for which values of a is this matrix *singular*?

Solution First of all, if $a = 0$ the matrix is already upper triangular and it has two pivots. So the matrix is singular for $a = 0$. Now, supposet $a \neq 0$. Then we can subtract the first row from the second and the third, thus getting

$$\begin{pmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & a-2 & a-3 \end{pmatrix}$$

As before, if $a = 2$ the matrix is upper triangular with only two pivots and so is singular. Let us now assume $a \neq 2$. So we can subtract the second from the third and get

$$\begin{pmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & 0 & a-4 \end{pmatrix}$$

This matrix is upper triangular and it is nonsingular if and only if a is not 0, 2 or 4. So those are the only values of a for which the matrix is singular.

1.2 Problem 2

Suppose we *already know* the inverse A^{-1} of a $m \times m$ matrix A. Now, we want to find the inverse $(A+uv^T)^{-1}$, where u and v are m-component column vectors. Ideally, we'd like to do this without re-doing the whole matrix-inversion process!

1.2.1 part (a)

Find the scalar (number) α so that

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{\alpha}$$

(Hint: if you see an expression like $x^T By$, realize that this is just a scalar and can be commuted with any other matrix/vector operations.)

Solution Multiplying both sides by $A + uv^T$ on the left we get

$$I = I - \frac{uv^T A^{-1}}{\alpha} + uv^T A^{-1} - \frac{uv^T A^{-1} uv^T A^{-1}}{\alpha} = I - \frac{uv^T A^{-1}}{\alpha} \left(\alpha - 1 - v^T A^{-1} u\right)$$

(remember that $v^T A^{-1} u$ is a number, and so it commutes with all matrices). So by choosing $\alpha = 1 + v^T A^{-1} u$ we have that the inverse of $A + uv^T$ is

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1} uv^T A^{-1}}{1 + v^T A^{-1} u}.$$

This is actually a famous relationship, known as the Sherman–Morrison formula, that is extremely useful in lots of real problems.

1.2.2 part (b)

Because matrix multiplication is associative, we can compute $A^{-1} uv^T A^{-1}$ from above in different orders:

$$A^{-1}(u(v^T A^{-1})) = A^{-1}((uv^T)A^{-1}) = (A^{-1}u)(v^T A^{-1})$$

If $m = 5$ (i.e. A is a 5×5 matrix and u and v are 5-component column vectors) compute *how many scalar multiplications* (multiplications of numbers) are required if we do the products indicated by the parentheses for these three different parenthesizations of $A^{-1} uv^T A^{-1}$, assuming you are given A^{-1}, u, and v (and that all matrix entries are nonzero so you can't skip any multiplies). (You *don't* need to actually do the matrix products, just work out how many multiplications they would require!)

Which order (parenthesization) would you choose to calculate $A^{-1} uv^T A^{-1}$ for your $(A + uv^T)^{-1}$ expression in part (a) in order to minimize your work?

For example, the outer product uv^T produces an $m \times m$ matrix, whose (i, j) entry is $u_i v_j$. So, there is one multiplication per entry of the output, or m^2 multiplications (25) in total to compute uv^T.

Solution To multiply a $k \times l$ matrix by a $l \times r$ matrix we need to do l multiplications for each entry of the result. So we need to do klr multiplications. Now let us do the three cases

- To compute $v^T A^{-1}$ we need to do m^2 multiplications and the result is an $m \times 1$ matrix. To compute $u(v^T A^{-1})$ we need to do m^2 additional multiplications and the result is an $m \times m$ matrix. Finally to compute $A^{-1}(u(v^T A^{-1}))$ we need to do m^3 multiplications. In total we did

$$m^2 + m^2 + m^3 = m^3 + 2m^2$$

  multiplications

- To compute uv^T we need to do m^2 multiplications. To compute $(uv^T)A^{-1}$ we need to do m^3 multiplications. Finally to compute $A^{-1}((uv^T)A^{-1}$ we need to do m^3 other multiplications. In total we need to do $2m^3 + m^2$ multiplications

- To compute $A^{-1}u$ we need to do m^2 multiplications and the result is an $m \times 1$ matrix. To compute $v^T A^{-1}$ we need to do m^2 multiplications and the result is an $1 \times m$ matrix. Finally to compute $(A^{-1}u)(v^T A^{-1})$ we need to do m^2 multiplications, so in total we did $3m^2$ multiplications.

The approach with the least amount of multiplications is by far the third one.

1.3 Problem 3

(Similar to Strang 2.6 problem 22.)

In pset 1, you did "upside-down" Gaussian elimination to convert the matrix

$$A = \begin{pmatrix} 1 & 6 & -3 \\ -2 & 3 & 4 \\ 1 & 0 & -2 \end{pmatrix}$$

into a *lower* triangular matrix

$$L = \begin{pmatrix} -0.5 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

Find an upper-triangular matrix U such that $A = UL$. (This example illustrates the fact that "upside-down" elimination corresponds to a "UL factorization" of A.)

Solution To find the matrix U we need to do in reverse order the operations we did to find L. So we need to start weith the identity

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then add twice the second row to the first

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and subtract twice the third row from the second and add $\frac{3}{2}$ times the third row to the first

$$U = \begin{pmatrix} 1 & 2 & 3/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

As we observed in class, the entries of U above the diagonal are just the multipliers for the row reduction with the opposite sign.

1.4 Problem 4

1.4.1 part (a)

Show that by multiplying a lower-triangular L matrix by a permutation (re-ordering) matrix P on the left *and* right you can convert L to an upper-triangular matrix PLP. You don't have to prove it in general, just find the matrix P that works for *any* 3×3 matrix L.

Once you have figured it out, check it. Enter your matrix P in Julia below, and use it to flip the following lower-triangular matrix to an upper-triangular one:

```
In [1]: X = [ 1  0  0
              2  3  0
              1  3 -1]
```

```
Out[1]: 3x3 Array{Int64,2}:
         1  0   0
         2  3   0
         1  3  -1
```

```
In [2]: P = [ 0 0 1
              0 1 0
              1 0 0]
```

```
Out[2]: 3x3 Array{Int64,2}:
         0  0  1
         0  1  0
         1  0  0
```

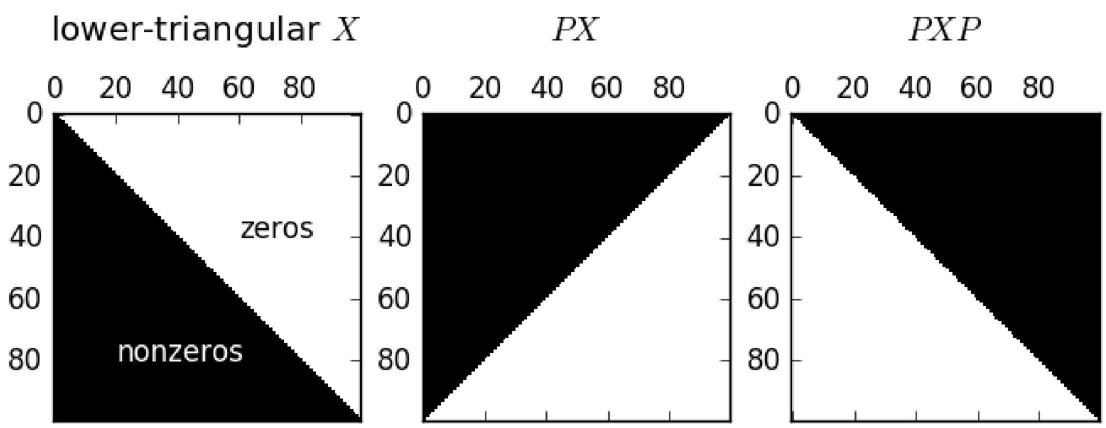
```
In [3]: P*X*P  # the result of this should be upper-triangular:
```

```
Out[3]: 3x3 Array{Int64,2}:
         -1  3  1
          0  3  2
          0  0  1
```

Solution Choose the matrix $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then multiplying by P on the left exchanges the first and third row, and multiplying by P on the right exchanges the first and third columns, so doing both will convert a lower triangular matrix to an upper triangular matrix and vice versa.

Although you were not required to do anything but the 3×3 case, this is one of those cases where the general case is almost easier to understand. If you have an arbitrary $m \times m$ lower triangular matrix, P should just be the permutation that **reverses the order** of the rows or columns. This is easy to see graphically:

```
In [4]: Xbig = Matrix(LowerTriangular(rand(100,100))) # a random 100×100 lower-triangular matrix
        Pbig = eye(100,100)[100:-1:1,:] # the permutation matrix that reverses the rows (or cols)
        using PyPlot
        subplot(1,3,1)
        spy(Xbig) # plot the nonzero pattern
        title(L"lower-triangular $X$", y=1.2)
        text(60,40, "zeros")
        text(20,80, "nonzeros", color="white")
        subplot(1,3,2)
        spy(Pbig*Xbig)
        title(L"PX", y=1.2)
        subplot(1,3,3)
        spy(Pbig*Xbig*Pbig)
        title(L"PXP", y=1.2)
```




```
Out[4]: PyObject <matplotlib.text.Text object at 0x31b843190>
```

If P is the reversal permutation, going from X to PX just reverses the rows, which flips the black triangle of nonzeros upside down but with the wrong orientation. Reversing *both* rows and columns with PXP gives the desired upper-triangular nonzero pattern.

1.4.2 part (b)

What is P^{-1}?

You can find it numerically from Julia with the command `inv(P)`, but you should still explain *why* it comes out that way:

```
In [5]: inv(P) # compute P^-1 numerically, using the P defined above
```

```
Out[5]: 3x3 Array{Float64,2}:
         -0.0  -0.0  1.0
         -0.0   1.0  0.0
          1.0   0.0  0.0
```

Solution We can see from above that `inv(P)` in Julia is in fact the same matrix as P (up to roundoff errors).

That is, $P^{-1} = P$. The reason for this is that P is the permutation that reverses the order of the rows, and reversing the order twice returns things back to the original order. Hence $P^2 = I$, that is $P^{-1} = P$.

It is also true that $P^{-1} = P^T$ for *any* permutation matrix, as explained in class, but this particular P is even more special than that.

1.4.3 part (c)

Suppose we take the A matrix from problem 3 and the P matrix from above, and compute the LU factorization $PAP = L'U'$ without row swaps (labeling the matrices L' and U' so that they aren't confused with the ones above), then compute $PL'P$ and $PU'P$. How to the results compare to your $A = UL$ factorization from problem 3?

Why? (You should be able to do some matrix algebra to turn $PAP = L'U'$ into $A = UL$.)

```
In [6]: A = [ 1  6 -3
             -2  3  4
              1  0 -2 ]
        L′, U′ = lu(P*A*P, Val{false}) # LU factorization of PAP without row swaps
```

```
Out[6]: (
        3x3 Array{Float64,2}:
          1.0  0.0  0.0
         -2.0  1.0  0.0
          1.5  2.0  1.0,

        3x3 Array{Float64,2}:
         -2.0  0.0   1.0
          0.0  3.0   0.0
          0.0  0.0  -0.5,

        [1,2,3])
```

```
In [7]: P*L′*P
```

```
Out[7]: 3x3 Array{Float64,2}:
         1.0  2.0   1.5
         0.0  1.0  -2.0
         0.0  0.0   1.0
```

```
In [8]: P*U′*P
```

```
Out[8]: 3x3 Array{Float64,2}:
         -0.5  0.0   0.0
          0.0  3.0   0.0
          1.0  0.0  -2.0
```

Solution Numerically, you should notice that `P*L′*P` is *exactly* the U matrix from our $A = UL$ factorization in problem 3, and `P*U′*P` is exactly the L matrix. Now we want to *derive* this algebraically.

We want to transform the equation $PAP = L'U'$ into something of the form $A = \cdots$. So we multiply on the left and the right by $P^{-1} = P$ and we obtain

$$A = PL'U'P$$

Now, we want to see the right hand side as the product of an upper triangular matrix by a lower triangular matrix. From part (a) we know that $PL'P$ and $PU'P$ are of the right form. Moreover, since $P^2 = I$ we have that

$$A = PL'U'P = PL'P^2U'P = (PL'P)(PU'P)$$

Finally, since P just permutes rows and columns, the diagonal entries of $PL'P$ are all 1s, exactly as expected for an elimination matrix (as problem 3, above). So $(PL'P)(PU'P) = UL$ gives $U = PL'P$ and $L = PU'P$ from the UL factorization of A.