1 Problem Set 2

Due Wednesday, 2/22 at 11am.

1.1 Problem 1

(From Strang, section 2.2, problem 25.)

\[ A = \begin{pmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{pmatrix} \]

For which three numbers \( a \) will elimination fail to give three pivots for this matrix? That is, for which values of \( a \) is this matrix singular?

1.2 Problem 2

Suppose we already know the inverse \( A^{-1} \) of a \( m \times m \) matrix \( A \). Now, we want to find the inverse \( (A + uv^T)^{-1} \), where \( u \) and \( v \) are \( m \)-component column vectors. Ideally, we’d like to do this without re-doing the whole matrix-inversion process!

1.2.1 part (a)

Find the scalar (number) \( \alpha \) so that

\[ (A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{\alpha} \]

(Hint: if you see an expression like \( x^TBy \), realize that this is just a scalar and can be commuted with any other matrix/vector operations.)

1.2.2 part (b)

Because matrix multiplication is associative, we can compute \( A^{-1}uv^TA^{-1} \) from above in different orders:

\[ A^{-1}(u(v^TA^{-1})) = A^{-1}((uv^T)A^{-1}) = (A^{-1}u)(v^TA^{-1}) \]

If \( m = 5 \) (i.e. \( A \) is a \( 5 \times 5 \) matrix and \( u \) and \( v \) are \( 5 \)-component column vectors) compute how many scalar multiplications (multiplications of numbers) are required if we do the products indicated by the parentheses for these three different parenthesizations of \( A^{-1}uv^TA^{-1} \), assuming you are given \( A^{-1} \), \( u \), and \( v \) (and that all matrix entries are nonzero so you can’t skip any multiplies). (You don’t need to actually do the matrix products, just work out how many multiplications they would require!)

Which order (parenthesization) would you choose to calculate \( A^{-1}uv^TA^{-1} \) for your \( (A + uv^T)^{-1} \) expression in part (a) in order to minimize your work?

For example, the outer product \( uv^T \) produces an \( m \times m \) matrix, whose \((i,j)\) entry is \( u_i v_j \). So, there is one multiplication per entry of the output, or \( m^2 \) multiplications (25) in total to compute \( uv^T \).
1.3 Problem 3
(Similar to Strang 2.6 problem 22.)
In pset 1, you did “upside-down” Gaussian elimination to convert the matrix

\[
A = \begin{pmatrix} 1 & 6 & -3 \\ -2 & 3 & 4 \\ 1 & 0 & -2 \end{pmatrix}
\]

into a lower triangular matrix

\[
L = \begin{pmatrix} -0.5 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & -2 \end{pmatrix}
\]

Find an upper-triangular matrix \( U \) such that \( A = UL \). (This example illustrates the fact that “upside-down” elimination corresponds to a “UL factorization” of \( A \).)

1.4 Problem 4
1.4.1 part (a)
Show that by multiplying a lower-triangular \( L \) matrix by a permutation (re-ordering) matrix \( P \) on the left and right you can convert \( L \) to an upper-triangular matrix \( PLP \). You don’t have to prove it in general, just find the matrix \( P \) that works for any \( 3 \times 3 \) matrix \( L \).

Once you have figured it out, check it. Enter your matrix \( P \) in Julia below, and use it to flip the following lower-triangular matrix to an upper-triangular one:

In [ ]: \( X = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 3 & -1 \end{pmatrix} \)

In [ ]: \( P = [ \ldots \text{enter your matrix here} \ldots ] \)

In [ ]: \( P \times X \times P \) # the result of this should be upper-triangular:

1.4.2 part (b)
What is \( P^{-1} \)?
You can find it numerically from Julia with the command \( \text{inv}(P) \), but you should still explain why it comes out that way:

In [ ]: \( \text{inv}(P) \) # compute \( P^{-1} \) numerically

1.4.3 part (c)
Suppose we take the \( A \) matrix from problem 3 and the \( P \) matrix from above, and compute the LU factorization \( PAP = L'U' \) without row swaps (labeling the matrices \( L' \) and \( U' \) so that they aren’t confused with the ones above), then compute \( PL'P \) and \( PU'P \). How to the results compare to your \( A = UL \) factorization from problem 3?

Why? (You should be able to do some matrix algebra to turn \( PAP = L'U' \) into \( A = UL \).)

In [ ]: \( A = \begin{pmatrix} 1 & 6 & -3 \\ -2 & 3 & 4 \\ 1 & 0 & -2 \end{pmatrix} \)

\( \text{L}, \text{U} = \text{lu}(P \times A \times P, \text{Val{false}}) \) # LU factorization of \( PAP \) without row swaps

In [ ]: \( P \times \text{L} \times P \)

In [ ]: \( P \times \text{U} \times P \)