# pset3-sol

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# 1 18.06 pset 3 Solutions

## 1.1 Problem 1

Suppose that you solve $AX = B$ with

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and find that $X$ is

$$X = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

### 1.1.1 (a)

What is $A^{-1}$?

(You should not have to apply brute-force Gaussian elimination to invert any matrices, nor should you use Julia in this part. You should be able to show how to do this quickly *by hand*.)

(This is not because we care about hand calculation *per se*, but rather because it is useful to be able to recognize and exploit special structure in matrices, and to understand the relationship between solving systems of right-hand-sides and finding $A^{-1}$.)

### 1.1.2 (b)

Evaluate a simple expression to *check* your answer from (a) by brute-force calculation in Julia.

For example, you can compute $B^{-1}X^{-1}$ by `inv(B) * inv(X)` in Julia. There should be some simple product of matrices or matrix inverses that gives $A^{-1}$. Figure it out!

```
In [1]: # here are the matrices B and X in Julia form
        B = [1 1 1 1
             0 2 2 2
             0 0 1 1
             0 0 0 1]
        X = [1 1 0 1
             0 0 1 0
             1 3 1 0
             2 0 0 1]
```

```
Out[1]: 4x4 Array{Int64,2}:
         1  1  0  1
         0  0  1  0
         1  3  1  0
         2  0  0  1
```

```
In [ ]: inv(B) * inv(X) ## FIX THIS: change to an expression that will give A^{-1}, and evaluate
```

#### 1.1.3 Solution

**(a)** Since $AX = B$, we have by inverting both sides that $X^{-1}A^{-1} = B^{-1}$. Then, by multiplying on the left by $X$ we finally get

$$A^{-1} = XB^{-1}$$

Equivalently, we could write $A = BX^{-1}$ (multiplying on the right by $X^{-1}$) and then invert both sides. Equivalently, we could multiply both sides of $AX = B$ on the left by $A^{-1}$ to get $X = A^{-1}B$, then moth both sides on the right by $B^{-1}$ to get $XB^{-1} = A^{-1}$.

So, to find $A^{-1}$ we must multiply $X$ by the inverse of $B$. Equivalently, we need to solve the system of equations $A^{-1}B = X$ for $A^{-1}$. One way to proceed is row-by-row: solve $a^TB = x^T$, where $a^T$ is each row of $A^{-1}$ and $x^T$ is the corresponding row of $x$; equivalently, solve $B^Ta = x$. Since $B$ is upper-triangular (hence $B^T$ is lower triangular), this is easy: we could just solve $B^Ta = x$ by forward substitution. For example, to get the *first row* of $A^{-1}$, we could solve

$$B^Ta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = x$$

From the first, row we get $a_1 = 1$. The second row gives $a_1 + 2a_2 = 1$, so $a_2 = 0$. The third row gives $a_1 + 2a_2 + a_3 = 0$, so $a_3 = -1$. The fourth row gives $a_1 + 2a_2 + a_3 + a_4 = 1$, so $a_4 = 1$. Hence the first row of $A^{-1}$ is $\begin{pmatrix} 1 & 0 & -1 & 1 \end{pmatrix}$. We would proceed similarly for the other rows of $A^{-1}$.

Alternatively, there is another way to proceed that is probably easier for this particular matrix. We have $AX = B$, but we really *want* to have $I$ on the right-hand side, since solving $AY = I$ would give $Y = A^{-1}$. Multiplying both sides on the *right* by $B^{-1}$ gives $Y = XB^{-1} = A^{-1}$ as above, but multiplying on the *right* corresponds to *column* operations, and in particular *the column operations that turn $B$ into $I$*. If we do the *same* column operations on $X$, that will give $A^{-1}$. The reason that this viewpoint is useful is that, if we look at $B$, it is easy to see that for *this* $B$ there are very simple column operations that turn it into the identity matrix. If we just take each column and subtract the preceding column, the matrix becomes *diagonal*, and to make it into $I$ we then just have to divide the second column by 2. Doing the *same* column operations to $X$ yields:

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -2 & -1 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$

**(b)** As described above, the expression is

```
In [2]: X*inv(B)

Out[2]: 4x4 Array{Float64,2}:
          1.0   0.0  -1.0   1.0
          0.0   0.0   1.0  -1.0
          1.0   1.0  -2.0  -1.0
          2.0  -1.0   0.0   1.0
```

## 1.2 Problem 2

Consider the vector space $\mathcal{M}$ of $m \times m$ real-valued matrices for some $m$, say $m = 4$. True or false (and provide a counter-example if *false*).

1. The symmetric matrices in $\mathcal{M}$ are a subspace (matrices with $A^T = A$).

2. The "skew-symmetric (also called"antisymmetric") matrices (those with $A^T = -A$) in $\mathcal{M}$ are a subspace.
3. The invertible matrices in $\mathcal{M}$ are a subspace.
4. The singular matrices in $\mathcal{M}$ are a subspace.

#### 1.2.1 Solution

1. True: In fact multiplying a symmetric matrix by a scalar doesn't change the symmetry, and the sum of symmetric matrices is symmetric (all the operations are done to entry separately
2. True: For the same reason as before, plus the identities $-(x+y) = -x - y$ and $-(\lambda x) = \lambda(-x)$.
3. False: The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $-A$ are invertible, but $A + (-A) = 0$ is not.
4. False: The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are singular but their sum is the identity matrix, which is invertible.

### 1.3 Problem 3

(Strang, section 3.2, problem 22.) If $AB = 0$ then the column space of $B$ is contained in the ______ of $A$. Why?

#### 1.3.1 Solution

If $AB = 0$ the it must be true that column space of $B$ is contained in the nullspace of $A$: $C(B) \subseteq N(A)$.

One way to see this is that for any $y \in C(B)$, we have $y = Bx$ for some $x$. But then $Ay = ABx = 0x = 0$, so $y \in N(A)$.

Another way to see this is that $AB$ just multiplies $A$ by each column of $B$, so $AB = 0$ means that *every column* of $B$ is in $N(A)$, hence $C(B)$ (linear combinations of the columns) must also be in $N(A)$.

### 1.4 Problem 4

(Strang, section 3.2, problem 29.) If $A$ is $4 \times 4$ and invertible, what is the nullspace of the $4 \times 8$ matrix $B = \begin{pmatrix} A & A \end{pmatrix}$?

#### 1.4.1 Solution

Write an 8-dimensional vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ where $x$ and $y$ are two 4-dimensional vectors. Then $v$ is in the nullspace of $B$ if and only if $Bv = 0$. But

$$Bv = \begin{pmatrix} A & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = Ax + Ay = A(x+y)$$

So $Bv = 0$ if and only if $A(x+y) = 0$. Since $A$ is invertible, this can happen if and only if $x + y = 0$, that is $x = -y$. So the nullspace of $B$ is composed by those vectors of the form $\begin{pmatrix} x \\ -x \end{pmatrix}$ if $x$ is a 4-dimensional vector.

Another way to see this is that if we convert $B$ to rref form, the same row operations that give $I$ in the first 4 columns (since $A$ is invertible the rank is 4) will also give $I$ in the last four columns, so we will get $R = \begin{pmatrix} I & I \end{pmatrix}$. From class, the null space is then simply the column space of

$$\begin{pmatrix} -I \\ I \end{pmatrix}$$

i.e. the columns of this matrix are a basis for $N(B)$, and these columns span all vectors of the form $\begin{pmatrix} x \\ -x \end{pmatrix}$.

## 1.5 Problem 5

(Strang, section 3.2, problem 23.) The reduced-row echelon form $R$ of a $3 \times 3$ matrix with randomly chosen entries is almost sure to be _________. What $R$ is virtually certain if the random matrix is $4 \times 3$?

### 1.5.1 Solution

The reduced row echelon form $R$ of a $3 \times 3$ matrix with randomly chosen real entries is almost sure to be the identity — the probability of getting an exact cancellation for one of the pivots is zero. If $R$ is $4 \times 3$, the reduced row echelon form will almost certainly still have 3 pivots, so will be of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

## 1.6 Problem 6

(Strang, section 3.2, problem 58.) Suppose $R$ is $m \times n$ of rank $r$, with pivot columns first:

$$R = \begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix}$$

where $I$ is an identity matrix and 0 denotes a block of zeros.

1. What are the shapes of those four blocks of $R$?
2. Find a *right-inverse* matrix $B$ such that $RB = I$ if $r = m$ (the zero blocks are gone).
3. What is the reduced-row echelon form of $R^T$?
4. What is the reduced-row echelon form of $R^T R$?

(In the last four parts, indicate both blocks like $I$ or 0 and their shapes.)

### 1.6.1 Solution

1. $I$ is an $r \times r$ block, so $F$ has to be an $r \times (n - r)$ and the two zero blocks are $(m - r) \times r$ and $(m - r) \times (n - r)$ respectively.

2. It suffices to choose $B = \begin{pmatrix} I \\ 0 \end{pmatrix}$ where the top block is an $r \times r$ identity matrix and the block of zeros is an $(n - r) \times r$ block.

3. The reduced row echelon form of $R^T$ is
$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$
where the identity is $r \times r$ and the other blocks are such that the whole matrix is $n \times m$. This is because
$$R^T = \begin{pmatrix} I & 0 \\ F^T & 0 \end{pmatrix}$$
and we can eliminate all the entries in $F^T$ via row operations.

4. The reduced row echelon form of $R^T R$ is
$$\begin{pmatrix} I & E \\ 0 & D \end{pmatrix}$$
where the identity block is $r \times r$, $E$ is an $r \times (n - r)$ block that is identical to $F$ except in the columns above pivots of $D$, where it is 0, the zero block has size $(n - r) \times r$ and $D$, of size $(n - r) \times (n - r)$, is

the reduced row echelon form of $-F^TF$. In fact, let $C$ be the lower triangular $(n-r)\times(n-r)$ matrix such thata $-CF^TF = D$ is the row echelon form of $-F^TF$. Then, since

$$R^TR = \begin{pmatrix} I & F \\ F^T & 0 \end{pmatrix}$$

and we can use row operations to realize the multiplication on the left by the lower triangular matrix

$$L = \begin{pmatrix} I & 0 \\ -CF^T & C \end{pmatrix}$$

and

$$LR^TR = \begin{pmatrix} I & F \\ 0 & -CF^TF \end{pmatrix} = \begin{pmatrix} I & F \\ 0 & D \end{pmatrix}$$

Finally we can do row operations to eliminate the columns of $F$ that lie above pivots of $D$.