pset4-sol

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1 18.06 Pset 4 Solutions

1.1 Problem 1

(Similar to Strang, section 3.2, problem 49.)

We showed in class that $C(AB) \subseteq C(A)$. Since the dimension of the column space is the rank, and a subspace always has a dimension $\leq$ the dimensionality of the enclosing space, this means that $\text{rank}(AB) \leq \text{rank}(A)$.

Using a similar reasoning, show that $\text{rank}(AB) \leq \text{rank}(B)$. Hint: consider the transpose $(AB)^T = B^T A^T$.

1.1.1 Solution

We are use the fact that A and A^T have the same rank. Then

$$\text{rank}(AB) = \text{rank}((AB)^T)) = \text{rank}(B^T A^T) \leq \text{rank}(B^T) = \text{rank}(B)$$

where $\text{rank}(B^T A^T) \leq \text{rank}(B^T)$ is follows from $\text{rank}(CD) \leq \text{rank}(C)$ above (changing the letters for clarity).

1.2 Problem 2

(Similar to Strang, section 3.4, problem 26 and 30.)

Find a basis (and the dimension) for each of these subspaces of 3×3 matrices:

- All diagonal matrices
- All symmetric matrices ($A^T = A$).
- All skew-symmetric (anti-symmetric) matrices ($A^T = -A$).
- All matrices whose nullspace contains the vector $(2, 1, -1)$.

1.2.1 Solution

Diagonal matrices Every diagonal matrix is of the form $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ and so can be written as a linear combination of the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since this system of generators is clearly minimal (none of these matrices is a linear combination of the others), this is a basis and the dimension of the space of diagonal matrices is 3.

Symmetric matrices Every symmetric matrix is of the form

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

and so can be written as a linear combination of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Again, since no matrix here is a linear combination of the others, this is a basis and the dimension of the space of symmetric matrices is 6.

Antisymmetric matrices Every antisymmetric matrix is of the form

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

and so can be written as a linear combination of

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Since these matrices form again a minimal set of generators, they are a basis. So the dimension of the space of antisymmetric matrices is 3.

Matrices such that $(2, 1, -1) \in N(A)$ A matrix A has $(2, 1, -1)$ in its nullspace if and only if

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2a + b - c \\ 2d + e - f \\ 2g - h + i \end{pmatrix} = 0$$

It is clear that we can find a basis for all such matrices easily if we can find a basis for the space of row vectors $\begin{pmatrix} x & y & z \end{pmatrix}$ such that $2x + y - z = 0$. Equivalently, we want a basis for the left nullspace of $(2, 1, -1)$, or the nullspace

$$N(\begin{pmatrix} 2 & 1 & -1 \end{pmatrix})$$

Since this matrix is essentially in row-reduced echelon form already (just multiplied by 2), we can apply our standard methods to find a nullspace basis, for example:

$$(1, -2, 0), \ (1, 0, 2)$$

So we can write the vector space of matrices such that $(2, 1, -1) \in N(A)$ as spanned by

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

So the dimension of this space is 6 (which is not surprising since we have put 3 conditions on a 9-dimensional space).

2 Problem 3

(Strang, section 3.5, problem 21.)

Suppose $A = uv^T + wz^T$ (it is the sum of two rank-1 matrices).

- Which vectors span the column space of A?
- Which vectors span the row space of A?
- The rank of A is less than 2 if ???????? or if ????????.
- Compute A and its rank if $u = z = (1, 0, 0)$ and $v = w = (0, 0, 1)$. Check your answer with Julia below.

In [1]:
```
u = z = [1,0,0]
v = w = [0,0,1]
A = u*v' + w*z'
```

Out[1]:
```
3x3 Array{Int64,2}:
 0  0  1
 0  0  0
 1  0  0
```

In [2]:
```
rank(A)
```

Out[2]:
```
2
```

2.0.1 Solution

- The vectors u and w span the column space, since every column is a linear combination of these two vectors. (They may not be a basis, because they may not be independent!)
- The vectors v and z span the row space, for the same reason. (Again, possibly not a basis.)
- Generically (for random vectors u, v, w, z), we would expect the rank of A to be 2. It will only be less than 2 in the following special cases:
- $A = 0$ (e.g. $u = w = 0$, $v = w = 0$, or other cases where $uv^T = -wz^T$). Then it is rank 0.
- If $A \neq 0$ and u and w are parallel (one is a multiple of the other) or v and z are parallel, then A will be rank 1. (In this case, it is easy to see that the column/row space is spanned by a single nonzero vector.)
- When $u = z = (1, 0, 0)^T$ and $v = w = (0, 0, 1)^T$, we obtain:

$$A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

  And A has rank 2, as expected. Our result is checked in Julia above.

3 Problem 4

(Based on Strang, section 4.1, problem 9.)

The following is an important property of the very important matrix A^TA (for real matrices) that will come up several times in 18.06:

- If $A^TAx = 0$ then $Ax = 0$. Reason: If $A^TAx = 0$, then Ax is in the nullspace of A^T and also in the ?????? of A, and those spaces are ???????. Conclusion: $N(A^TA) = N(A)$.
- Alternative proof: $A^TAx = 0$, then $x^TA^TAx = 0 = (Ax)^T(Ax)$. Why does this imply that $Ax = 0$? (Hint: if $y^Ty = 0$, can we have $y \neq 0$?)

- If A is a random $m \times n$ matrix, what can you conclude about the ranks of A^TA and AA^T? Try it in Julia for a 5×7 random matrix:

```
In [3]: A = randn(5,7)

Out[3]: 5x7 Array{Float64,2}:
         -1.42488    0.49731     0.193989  -0.518111  -1.60314   -1.79199   -0.803947
         -1.12835   -0.236285   -1.48037   -1.38106    0.278083   1.33166    1.48049
          0.595007  -0.0488672  -1.13801    0.330875   0.713253  -0.785896  -0.673687
         -0.394517   0.361648   -0.428049   0.364408  -0.504469  -0.525341  -1.83838
         -1.2217     1.40207    -0.230121  -0.168273   1.43475    0.475414   0.254754

In [4]: rank(A'*A)

Out[4]: 5

In [5]: rank(A*A')

Out[5]: 5
```

3.0.1 Solution

- If $A^TAx = 0$ then $Ax = 0$. Reason: If $A^TAx = 0$, then Ax is in the nullspace of A^T and also in the **column space** of A, and those spaces are **orthogonal.** So Ax is orthogonal to itself, thus $Ax = 0$. Conclusion: $N(A^TA) = N(A)$.
- If $(Ax)^T(Ax) = 0$ this means that the length of the vector Ax is 0. But the only vector with zero length is the zero vector, since a sum of squares can be zero only if all the squares are zero.
- In our 5×7 example A in Julia above, we saw that the ranks of A^TA and AA^T were 5; this is essentially certain to happen. Reason: if an $m \times n$ matrix A has rank r, then $N(A) = N(A^TA)$ has dimension $n - r$, and hence the $n \times n$ matrix A^TA also has rank r. The same reasoning can be applied to A^T and AA^T, and hence $\text{rank}(A) = \text{rank}(A^TA) = \text{rank}(AA^T)$. A random A is almost certainly full rank, so it will have $\text{rank}(A) = \min(m, n)$, and therefore A^TA and AA^T will have the same rank $\min(m, n)$ ($= 5$ in this case).