18.06 pset 7
Due Wednesday, April 5.

1.1 Problem 1
Refer to the orthogonal polynomials notebook from class for this problem.

In class, we defined an dot product $f \cdot g = \int_{-1}^{1} f(x)g(x)dx$ for functions on $x \in [-1,1]$, and using this we showed how we could apply Gram–Schmidt to the polynomials \{1, x, x^2, \ldots\} to find an orthogonal of polynomials $p_k(x)$, the Legendre polynomials.

1.1.1 part (a)
In class, I claimed that by performing the orthogonal projection of any function $f(x)$ onto these polynomials, we obtain the least-square fit polynomial on the interval $[-1,1]$. In this problem, you will apply basic calculus to show that explicitly.

Suppose we have an orthonormal basis $q_0(x), q_1(x), q_2(x), q_3(x)$ for all degree $\leq 3$ polynomials (the vector space $P_3$), i.e. $q_i \cdot q_j = 0$ if $i \neq j$ and $= 1$ if $i = j$, using our dot product from above. Given a real-valued function $f(x)$ on $[-1,1]$ (with finite $f \cdot f$ — none of these integrals should blow up!), we want to find the closest degree-3 polynomial to $f$ in the least-square sense:

$$\min_{p \in P_3} \int_{-1}^{1} [f(x) - p(x)]^2 dx = \min_{p \in P_3} (f - p) \cdot (f - p) = \min_{p \in P_3} \|f - p\|^2$$

Write $p(x)$ in our orthonormal basis:

$$p(x) = c_0 q_0(x) + c_1 q_1(x) + c_2 q_2(x) + c_3 q_3(x)$$

At the minimum $p$ (the least-square fit), basic calculus tells us that the partial derivative must be zero:

$$\frac{\partial}{\partial c_k} \|f - p\|^2 = 0$$

Show that this leads to the condition $c_k = q_k \cdot f$, which is exactly the coefficient of the orthogonal projection.

Hint: You can easily verify that the product rule $\frac{\partial}{\partial c} (f \cdot g) = \left( \frac{\partial f}{\partial c} \cdot g \right) + \left( f \cdot \frac{\partial g}{\partial c} \right)$ works for dot products of functions!

1.1.2 part (b)
Suppose that we have real-valued function $f(x)$ that is in the span of an infinite orthonormal basis $q_k(x)$ of functions (e.g. polynomials as above) on $[-1,1]$ with the dot product from above, i.e.

$$f(x) = \sum_{k=0}^{\infty} c_k q_k(x)$$
for coefficients \( c_k = q_k \cdot f \). Assuming \( \|f\| \) is finite (i.e. the function \( f \) is square-integrable), derive the identity:

\[
\|f\|^2 = f \cdot f = \sum_{k=0}^{\infty} c_k^2
\]

(This result is called Parseval’s theorem for Fourier series.)

**How does this relate to problem 4 of pset 6?**

(For people who have taken 18.100 or similar: assume you can freely interchange/re-order the infinite sums, limits, integrals, etcetera; doing this properly would involve establishing some technical conditions on the infinite series here.)

### 1.2 Problem 2

Apply Gram-Schmidt to the polynomials \(1, x, x^2\) to find an orthonormal basis of polynomials under the *different* dot product:

\[
f \cdot g = \int_0^{\infty} f(x)g(x)e^{-x}dx
\]

There are lots of ways to define dot products in practice, and in real applications the choice of dot product depends a lot on the problem you are solving. For example, one might want to weight the errors differently at different points (here, weighting by \(e^{-x}\)) in a least-square fit.

### 1.3 Problem 3

(Based on Strang, section 6.2, problem 33.) Consider the following four \(2 \times 2\) matrices, which have very similar-looking entries:

\[
\begin{bmatrix}
3 & 2 \\
1 & 4
\end{bmatrix}
\begin{bmatrix}
3 & 2 \\
-5 & -3
\end{bmatrix}
\begin{bmatrix}
5 & 7 \\
-3 & -4
\end{bmatrix}
\begin{bmatrix}
5 & 6.9 \\
-3 & -4
\end{bmatrix}
\]

1.3.1 (a)

Compute each matrix to the **100th power** in Julia, e.g. compute \(A^{100}\) in Julia by \(A^{100}\). The results should be very different!
1.3.2 (b)

All of these matrices are diagonalizable (can be written as $X \Lambda X^{-1}$ as in lecture), with two distinct eigenvalues $\lambda$. The function `eigvals(A)` computes the eigenvalues of $A$ in Julia. Using the built-in `eigvals` function, compute the eigenvalues of these four matrices, and use them to explain the results you observed in part (a).

Note that the eigenvalues may be complex numbers, even for real matrices, just as the roots of a real polynomial may be complex! The complex number $z = a + bi$ in Julia is written $z = a + b*im$. Complex numbers can also be written in polar form $z = re^{i\theta}$, where $r = \text{abs}(z)$ and $\theta = \text{angle}(z)$ in Julia. Recall that $z^n = r^ne^{i\theta}$ blows up if $|z| = r = \text{abs}(z)$ is $> 1$.

In []:

1.4 Problem 4
1.4.1 (a)

Based on Strang, section 5.1, problem 8. Prove that every orthogonal matrix ($Q^TQ = I$) has determinant $+1$ or $-1$, in two ways:

- Use the product rule $\det(AB) = (\det A)(\det B)$ and the transpose rule $\det Q = \det Q^T$.
- Use only the product rule. If $|\det Q| < 1$ then $\det Q^n = (\det Q)^n$ goes to zero: $Q^n$ becomes nearly singular for large $n$. How do you know that this can’t happen to $Q^n$?
- Hint: $(Q^n)^T(Q^n) = ???$ so $Q^n$ is $???$.
- Alternatively, think about problem 4 of pset 6, and note that a nearly singular matrix $A$ has a vector $x \neq 0$ that is nearly in a nullspace ($Ax$ is nearly zero).

1.4.2 (b)

If $\det A = 1$, does that mean that $A$ is orthogonal? Explain why or provide a counterexample if it is false.

1.4.3 (c)

If $\det A = 1234$, what is $\det R$ where $R$ is the upper-triangular matrix in the QR factorization of $A$?

1.5 Problem 5
1.5.1 (a)

The function $X = \text{randn}(5,5)$ in Julia generates a random $5 \times 5$ matrix. Given $X$, we can compute a new matrix $Y = \alpha X$ for some scalar $\alpha$ such that $\det Y = 1234$. What is $\alpha$?

In []: $X = \text{randn}(5,5)$

# to make things easier, I'll force det(X) to be positive by flipping the sign of the first column if needed
if $\det(X) < 0$
    X[:,1] = -$X[:,1]$
end
$\det(X)$

In []: $\alpha = ???$  # fill in this line!
$Y = \alpha * X$
$\det(Y)$  # this should give 1234 (+ small roundoff error)
1.5.2 (b)
Using your matrix $Y$, compute its QR factorization by $Q, R = qr(Y)$ and use this to check your answer from problem 4(c) above.

```python
In [ ]: Q, R = qr(Y)
```

```python
In [ ]:
```