MIT 18.06 Exam 1 **Solutions**, Spring 2022
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Problem 1 (26 points):

Suppose

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 5 \\ 1 & 2 & 1 & 1 \end{pmatrix}.$$

(a) Give a basis for $N(A)$:

We proceed by elimination to reduce A to upper-triangular form:

$$A = \begin{pmatrix} \boxed{1} & 2 & 1 & 2 \\ 2 & 4 & 2 & 5 \\ 1 & 2 & 1 & 1 \end{pmatrix} \overset{\substack{r_2 - 2r_1 \\ r_3 - r_1}}{\longrightarrow} \begin{pmatrix} \boxed{1} & 2 & 1 & 2 \\ & & & \boxed{1} \\ & & & -1 \end{pmatrix} \overset{r_3+r_1}{\longrightarrow} \begin{pmatrix} \boxed{1} & 2 & 1 & 2 \\ & & & \boxed{1} \\ & & & 0 \end{pmatrix} = U,$$

which immediately tells us that A is rank 2, and that the 1st and 4th columns are the pivot columns. We will then solve equations by dividing the variables into pivot and free variables, $x = [p_1, f_1, f_2, p_2]$. The nullspace will therefore be $4 - 2 = 2$ dimensional, and our basis will need **2 vectors**.

To find our usual basis for $N(A)$, the special solutions, we will set the free variables to $[1, 0]$ and $[0, 1]$ and solve for the pivot variables, which leads to the upper-triangular systems:

$$\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

where the left-hand-side is the upper-triangular matrix in the pivot columns of U and the right-hand-side is minus the free columns. By backsubstitution, we get $p_2 = 0$ in both cases and $p_1 = -2$ or -1, respectively. Plugging these into $x = [p_1, f_1, f_2, p_2]$, we get our "special" basis for $N(A)$:

$$N(A) = \text{span of} \boxed{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}.$$

(b) For what value or values (if any) of α does $Ax = \begin{pmatrix} 1 \\ 2\alpha \\ \alpha \end{pmatrix}$ have any solution x?

To check whether a solution exists, we apply the same elimination steps from $A \to U$ to this right-hand-side, and check if it is zero in the 3rd row (matching the row of zeros in U), which is ensures that it is in $C(A)$. Hence:

$$\begin{pmatrix} 1 \\ 2\alpha \\ \alpha \end{pmatrix} \overset{\substack{r_2 - 2r_1 \\ r_3 - r_1}}{\longrightarrow} \begin{pmatrix} 1 \\ 2\alpha - 2 \\ \alpha - 1 \end{pmatrix} \overset{r_3 + r_1}{\longrightarrow} \begin{pmatrix} 1 \\ 2\alpha - 2 \\ 3\alpha - 3 \end{pmatrix},$$

giving the condition $3\alpha - 3 = 0$, i.e. $\boxed{\alpha = 1}$.

Problem 2 (24 points):

Give a **basis** for the **nullspace** $N(A)$ and a basis for the **column space** $C(A)$ for each of the following matrices:

(a) The one-column matrix $A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

This matrix is obviously rank 1 (full column rank), so $N(A) = \{\vec{0}\}$ and the basis for $N(A)$ is the **empty set** $\{\}$: the nullspace is zero-dimensional so it needs *no* basis vectors. A basis for $C(A)$ is just $\boxed{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}}$, the first column of A (which is also the pivot column).

(b) The one-row matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$.

This matrix is also rank 1 (full row rank), with 1 pivot column and 3 free columns. We can read off the special solutions, so the 3-dimensional nullspace $N(A)$ has the basis

$$\boxed{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}}.$$

More explicitly, the special solutions are of the form (p_1, f_1, f_2, f_3), where we set the free variables to $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ (the columns of I) and solve for p_1, but since this is one equation in one variable we can do it by inspection: p_1 is just equal minus the free column.

Since it has full row rank, the column space $C(A)$ is all of $\mathbb{R}^1$, and is spanned by the pivot column $\boxed{\begin{pmatrix} 1 \end{pmatrix}}$.

Note that in 18.06 we sometimes gloss over the distinction between $\mathbb{R}$ (scalars) and $\mathbb{R}^1$ (1-component column vectors) and $\mathbb{R}^{1\times 1}$ (1×1 matrices). If you think of A here as a "row vector" or "covector" that takes dot products with $[1,2,3,4]$, then the output is in $\mathbb{R}$ rather than $\mathbb{R}^1$ and you might say that a basis is the number $\boxed{1}$. I will accept that answer as well.

(c) The 100-row matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & 4 \end{pmatrix}$ in which every row is $\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$.

This also has rank 1—after elimination, all the rows after the first will be zero. So $N(A)$ will be 3-dimensional and $C(A)$ will be 1-dimensional.

The first thing to realize is that we are doing the same operation as in part (b), but we are *repeating the output* 100 times. This *doesn't change the nullspace*, since if the first row of the output is zero then all of the rows are zero. So the nullspace basis is the same as in part (b), i.e. $N(A)$ is spanned by the special solutions

$$\boxed{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}}.$$

The column space $C(A)$ is spanned by the pivot column—the first column, here—of A, which is simply

$$\boxed{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{100}},$$

i.e. **100 rows of 1's**.

Problem 3 (25 points):

Suppose that we are solving $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. In each of the parts below, a **complete** solution x is proposed. For each possibility, say **impossible** if that could *not* be a *complete* solution to such an equation, **or** give the the **size** $m \times n$ and the **rank** of the matrix A if x is possible.

(a) $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

Impossible. A would need to be a 3×4 matrix, but such a matrix would have rank ≤ 3 and hence could not have unique solutions (could not be full column rank).

(b) $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \alpha_1 \begin{pmatrix} 1 \\ -1 \\ 5 \\ 17 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ for all real numbers $\alpha_1, \alpha_2 \in \mathbb{R}$

Possible. Awould need to be a 3×4 matrix of rank 2, in order to have a 2d nullspace.

(c) $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for all real numbers $\alpha \in \mathbb{R}$

Impossible. For $\alpha = -1$, this would give $\vec{x} = \vec{0}$, which could not be a solution with a nonzero right-hand side.

(d) $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for all real numbers $\alpha \in \mathbb{R}$

Possible. A would need to be a 3×2 matrix of rank 1, in order to have a 1d nullspace.

(e) $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for all real numbers $\alpha_1, \alpha_2 \in \mathbb{R}$

Possible: 3×2 of rank 1. Since the second two vectors are the same, α_2 is redundant with α_1 and this is equivalent to the previous part with $\alpha = \alpha_1 + \alpha_2$.

Note: There was a typographical error in this problem: I had *meant* to make the two vectors linearly independent, i.e. to ask something like $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In this case the solution would have been **impossible**: A would need to be a 3×2 matrix, but to have a 2d nullspace it would need to have rank 0, which means that no non-zero right-hand-side could have a solution. Equivalently, the second two vectors form a basis for $\mathbb{R}^2$ if they are linearly independent, so there is some value of α_1 and α_2 that cancels the $(1, 2)$ vector and gives $\vec{x} = \vec{0}$, which cannot be a solution with a nonzero right-hand side.

Problem 4 (25 points):

Let

$$B = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & -1 & -1 \\ & 2 & -1 \\ & & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 5 \\ -8 \\ -4 \end{pmatrix}.$$

Compute:

$$(CB)^{-1}b.$$

(Hint: Remember what I said in class about inverting matrices!)

Solution: As usual, we don't want to compute matrix inverses explicitly, we want to solve linear systems. In this case

$$(CB)^{-1}b = \underbrace{B^{-1}\underbrace{C^{-1}b}_{y}}_{x},$$

where $y = C^{-1}b$ is computed by solving $Cy = b$ using **backsubstitution** (since C is upper-triangular), and then $x = B^{-1}y$ is computed by solving $Bx = y$ using **forward-substitution** (since B is lower-triangular). No Gaussian elimination is required! Proceeding, we have

$$\underbrace{\begin{pmatrix} 2 & -1 & -1 \\ & 2 & -1 \\ & & 2 \end{pmatrix}}_{C}\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} 5 \\ -8 \\ -4 \end{pmatrix}}_{b} \implies \begin{matrix} 2y_1 = 5 + y_2 + y_3 = -2 \implies y_1 = -1 \\ 2y_2 = -8 + y_3 = -10 \implies y_2 = -5 \\ 2y_3 = -4 \implies y_3 = -2 \end{matrix},$$

and hence

$$\underbrace{\begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{pmatrix}}_{B}\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} -1 \\ -5 \\ -2 \end{pmatrix}}_{y} \implies \begin{matrix} x_1 = -1 \\ x_2 = -5 - x_1 = -4 \\ x_3 = -2 - x_1 - x_2 = 3 \end{matrix},$$

giving us our solution

$$x = (CB)^{-1}b = \boxed{\begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}}.$$

Alternative solutions: You could, of course, solve this in other ways. You could multiply CB together to obtain $CB = \begin{pmatrix} 0 & -2 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{pmatrix}$, and then laboriously invert this, e.g. with Gauss–Jordan, to obtain $(CB)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{3}{8} \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} \\ 0 & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$, and finally multiply this by b. But this is a *lot* more work than doing a single backsolve followed by a single forward-solve, and is much more error-prone. In class, I repeatedly emphasized that you **almost never need to compute matrix inverses explicitly**, and if you do so then you are probably making a mistake. Inverses are useful for algebraic manipulations, but when it comes time to finally *calculate* something you should read them as "solving a linear system." Another way of viewing this is that, for $n \times n$ matrices, back/forward solves take $\sim n^2$ operations, but both multiplying CB and inverting the matrix take $\sim n^3$ operations, so if I had given you a larger matrix then the penalty of doing it the slow way would have been even more dramatic.