18.06 FINAL

(Note: those of you starting at 9+x hours, please add x hours to all the times below)

- *8:45 - exam available on Canvas (email jhahn01@mit.edu ASAP if you can't access it)*
- *9:00 - you may start reading the problems and working on them*
- *12:00 - you must stop writing at this time*
- *12:30 - Gradescope upload window will close. If you can't upload your scanned exam to Gradescope for technical reasons, please email it to jhahn01@mit.edu before this time*

NO collaboration or written/electronic/online sources allowed, except for our course materials (lecture and recitation notes, problem sets and review session problems + solutions).

DOWNLOAD or **DOWNLOAD + PRINT** this exam before the official start time. You can either annotate the PDF file, or physically write on paper (if you need extra sheets of paper, please write your name and the problem number on **ALL** pages that you want graded).

UPLOAD or **SCAN + UPLOAD** your exam to Gradescope after time is up and pencils are down. Make sure that the pages are easily legible (e.g. good camera quality and angle).

You **MUST** show your work to receive credit. **JUSTIFY EVERYTHING**. Just giving a correct answer without an explanation of what led you to it will be **SEVERELY** penalized.

There are $\boxed{7}$ problems, each worth $\boxed{50}$ points.

NAME: ____

MIT ID NUMBER: ____

RECITATION INSTRUCTOR: ____

PROBLEM 1 **NAME:** ____________________

(1) Compute the determinant of $\begin{bmatrix} -1 & 2 & 5 & -3 \\ 1 & -3 & -3 & 2 \\ -2 & 7 & 8 & -5 \\ -4 & 7 & 2 & -1 \end{bmatrix}$ by row reduction/product of pivots.

Show all the steps. *(10 pts)*

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(2) Compute the determinant of $\begin{bmatrix} 0 & -2 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 4 & 0 & 0 & 5 & 0 \\ 0 & 0 & 7 & 3 & 0 \\ 0 & 0 & 3 & 0 & -1 \end{bmatrix}$ <u>by cofactor expansion</u>.

Show all the steps. *(10 pts)*

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For the remainder of this problem, consider the following 10×10 matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(3) This matrix has two eigenvalues: just by inspecting the matrix, eyeball/guess what they are. Characterize the subspaces of eigenvectors corresponding to each eigenvalue. *(10 pts)*

Characterizing each of the two subspaces above can be done by either giving a basis of it, or by describing it implicitly as "the subspace of vectors satisfying the equation blah".

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(4) What are the geometric multiplicities of the eigenvalues of the matrix A from the previous part? What are the algebraic multiplicities? What is the characteristic polynomial?

Explain how you know. *(15 pts)*

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(5) Based on the previous parts, what is the determinant of the matrix:

$$B = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Explain how you know. *(5 pts)*

PROBLEM 2 **NAME:** ____________________

Throughout this problem, the matrix A has the following Singular Value Decomposition:

$$A = \underbrace{\frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ x & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\frac{1}{5}\begin{bmatrix} 4 & -3 \\ 3 & y \end{bmatrix}}_{V^T}$$

where the matrices U and V are orthogonal and x, y denote two mystery real numbers.

The matrices U and V include the prefactors $\frac{1}{3}$ and $\frac{1}{5}$, so the top-left entry of U is $\frac{2}{3}$ and the top-left entry of V is $\frac{4}{5}$; recall that orthogonal means that $U^T U = I_3$ and $V^T V = I_2$.

(1) What are the values of x, y based on the information given? **Explain how you know**.
(5 pts)

(2) Fill in the blanks (no explanation needed):

- the rank of the matrix A is ________ *(5 pts)*
- the eigenvalues of $A^T A$ are __________, and those of AA^T are __________ *(5 pts)*
- a non-zero eigenvector of AA^T is ________ (any one eigenvector will suffice) *(5 pts)*

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(3) Write A as a sum of two rank 1 matrices. *(5 pts)*

It suffices to write these rank 1 matrices as a column times a scalar times a row, namely $\mathbf{u} \cdot \sigma \cdot \mathbf{v}^T$*; you don't need to explicitly multiply the column, scalar and row out.*

(4) Without computing A out explicitly, calculate the vector:

$$A \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Explain how you know. *(5 pts)*

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(5) What is the maximum of the following quantity as $\mathbf{v}$ ranges over non-zero vectors in $\mathbb{R}^2$:

$$\frac{||A\mathbf{v}||}{||\mathbf{v}||}$$

and for what $\mathbf{v}$ is it achieved? *(10 pts)*

Here $||\mathbf{v}||$ denotes the length of the vector $\mathbf{v}$. Note that there is a whole line of $\mathbf{v}$'s for which the maximum is achieved; you simply need to find one non-zero vector on this line.

(6) Compute the pseudo-inverse A^+ of A, and explain how you got it (your answer for A^+ should be a 2×3 matrix with explicit numbers as entries). *(5 pts)*

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(7) Use A^+ to compute a least squares solution to $A\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (i.e. you must find a vector $\mathbf{v} \in \mathbb{R}^2$ such that $A\mathbf{v}$ is as close as possible to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; explain which formula you are using).

(5 pts)

PROBLEM 3 **NAME:** ____________________

In this problem, we will consider the matrix:

$$X = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & \alpha & 0 \end{bmatrix}$$

for some mystery real number α.

(1) What does α need to be so that the matrix has rank 2? How do you know? *(10 pts)*

(2) For the specific value of α from part (1), write down bases for the column space and the row space of X. (*Don't forget that there are as many vectors in a basis as the rank*) *(10 pts)*

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(3) Use the Gram-Schmidt process, and your result from part (2), to write down bases for:

- the left nullspace of X
- the nullspace of X

(α is still the specific number from part (1), which ensures that X has rank 2). *(20 pts)*

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(4) Compute the general solution of the system of equations:

$$X\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

where α is still the specific number from part (1), which ensures that X has rank 2. *(10 pts)*

PROBLEM 4 **NAME:** ____________________

(1) Write the complex number i in polar form:

$$i = re^{i\theta}$$

where r is a positive real number, and θ is an angle. **Explain your answer**. *(5 pts)*

(2) Write the integer powers i^k (for <u>all</u> $k \in \mathbb{Z}$) both in:

- Cartesian form $a + bi$;
- polar form $re^{i\theta}$.

Explain your answer. *(10 pts)*

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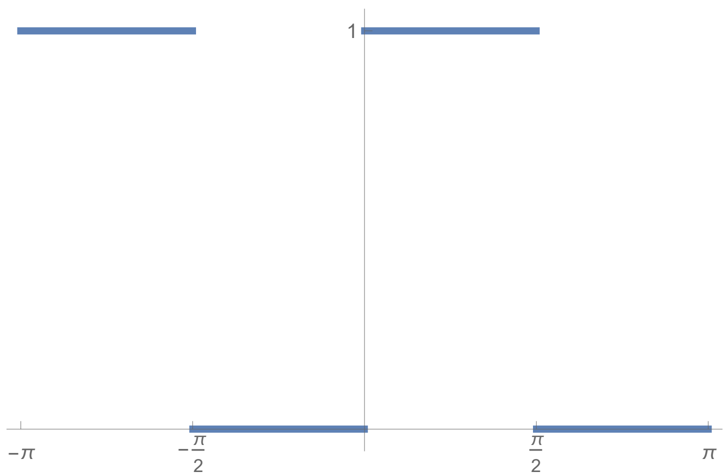
Consider the following "square" wave function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \left[-\pi, -\frac{\pi}{2}\right) \text{ or } x \in \left[0, \frac{\pi}{2}\right) \\ 0 & \text{if } x \in \left[-\frac{\pi}{2}, 0\right) \text{ or } x \in \left[\frac{\pi}{2}, \pi\right) \end{cases}$$

and extended to all $x \in \mathbb{R}$ by periodicity: $f(x + 2\pi) = f(x)$.



(3) Compute its <u>complex</u> Fourier series:

(∗)
$$f(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx}$$

by which I mean, compute the coefficients c_k. **Simplify as much as possible**. *(10 pts)*

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(4) Compute the real Fourier series of f (using sines and cosines) by applying the formula:

$$e^{ix} = \cos x + i \sin x$$

to the right hand side of formula $(*)$ in part (3). **Simplify as much as possible**. *(10 pts)*

(5) Using parts (1)-(4), compute the value of the Fourier series from $(*)$ at $x = \frac{\pi}{2}$, i.e.:

$$\sum_{k \in \mathbb{Z}} c_k e^{ik\frac{\pi}{2}}$$

where c_k are the coefficients you computed in part (1). *(5 pts)*

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(6) How does the value of the Fourier series at $x = \frac{\pi}{2}$ compare to the value of $f\left(\frac{\pi}{2}\right)$? How do you explain this? *(10 pts)*

PROBLEM 5 **NAME:** ____________________

Throughout this problem, we will work with the matrix:

$$Z = \begin{bmatrix} 2 & -1 & -1 \\ 6 & -2 & -4 \\ 1 & -1 & 0 \end{bmatrix}$$

(1) Compute the characteristic polynomial of Z, and work out the eigenvalues. *(15 pts)*

Points will be taken off if you use the diagonals' method (a.k.a. Sarrus' rule) to compute 3×3 *determinants. Use row reduction, cofactor expansion, or the big formula instead, and explain your process in detail.*

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(2) Diagonalize Z, i.e. write it as:

$$\boxed{Z = VDV^{-1}}$$

for an invertible matrix V and a diagonal matrix D. *(15 pts)*

Important for the following parts*: the diagonal entries of D should be the eigenvalues of Z, in order from the largest to the smallest.*

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(3) For a general 3×3 matrix:

$$Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}$$

find matrices M and N such that:

$$MY = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \qquad \text{and} \qquad YN = \begin{bmatrix} y_{11} & y_{13} \\ y_{21} & y_{23} \\ y_{31} & y_{33} \end{bmatrix}$$

No explanation needed here. Pay attention to the indices! *(10 pts)*

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(4) Using the previous parts of the problem, find 2×2 submatrices A, B and C of the 3×3 matrices V, D and V^{-1} (respectively) such that:

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = ABC$$

"Submatrix" means that, for instance, A is obtained from V by removing one row and one column (you do not need to write A explicitly, but make sure you say which row/column you need to remove from V to get it; **explain your reasoning**) *(10 pts)*

Hint: look at the 2×2 *matrix* $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ *in relation to* Z.

PROBLEM 6 **NAME:** ____________________

This problem consists of two parts: probability and statistics. While the language describing these two situations is often the same, the linear algebra tools used are different. So consider the two as different problems from a mathematical standpoint.

PROBABILITY: Paul the Octopus is not only good at predicting soccer scores, but he has magic powers. In his tank there are two boxes, one with the Dutch flag and one with the Spanish flag. Every time Paul creeps into one of the boxes, the respective soccer team scores a goal. However, for some reason Paul is k times more likely to visit the Spanish box than the Dutch box, for some fixed natural number k.

(1) What is, as a function of k, the probability that any one of Paul's visits is to the Dutch box? Same question for the Spanish box. *(5 pts)*

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(2) Now suppose that Paul makes two consecutive (independent) visits to the boxes, according to the rule above. Consider the random vector:

$$\mathbf{v} = \begin{bmatrix} s \\ d \end{bmatrix}$$

where s (respectively d) keeps track of how many goals the Spanish (respectively Dutch) team scored as a consequence of these two visits. What are the possible values for the vector $\mathbf{v}$ and what are their probabilities? *(5 pts)*

(3) Compute the mean (i.e. the expected value) of the random vector $\mathbf{v}$. *(5 pts)*

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(4) Compute the covariance matrix of the random vector $\mathbf{v}$. *(10 pts)*

(5) Diagonalize the covariance matrix computed in the previous part. Because the eigenvalues and eigenvectors are simple, you are allowed to simply guess them.

(10 pts)

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(6) One of the eigenvalues of the covariance matrix should be 0. If:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

is the corresponding eigenvector, then the random variable $v_1 \cdot s + v_2 \cdot d$ should have variance 0. How do you explain this intuitively (i.e. based on the original probability setup)? *(5 pts)*

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STATISTICS: Suppose you have m sets consisting of n samples each. You may think of these sets as vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, and collect them as the columns of an $n \times m$ matrix:

$$\boldsymbol{A} = \left[\ \boldsymbol{a}_1 \ \middle| \ \ldots \ \middle| \ \boldsymbol{a}_m \ \right] = \begin{bmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nm} \end{bmatrix}$$

The goal of principal component analysis (PCA) is to diagonalize the covariance matrix:

$$\boxed{K = \frac{\boldsymbol{A}^T P \boldsymbol{A}}{n-1} = QDQ^T}$$

where $P = \frac{1}{n} \begin{bmatrix} n-1 & -1 & \ldots & -1 \\ -1 & n-1 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & n-1 \end{bmatrix} = I_n - \frac{1}{n} \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}$.

(7) Show that $P^2 = P$ and $P^T = P$. *(5 pts)*

(you may give either a geometric argument, or one via algebraic manipulations with matrices)

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(8) Suppose you had a machine which takes in any matrix, and produces the matrices U, Σ and V that give its singular value decomposition $U\Sigma V^T$. How can you use this machine to obtain the orthogonal matrix Q in the boxed formula on the previous page? *(5 pts)*

PROBLEM 7 **NAME:** ____________________

(1) Fill in the blanks. The orthogonal complement of the vector:

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

consists of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which satisfy the equation ________________ = ______ *(5 pts)*

(2) Choose basis vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ of the orthogonal complement of $\boldsymbol{a}$, and explain why $\boldsymbol{b}$, $\boldsymbol{c}$ form a basis. *Hint: your future will be easier if you pick* $\mathbf{b}$ *and* $\mathbf{c}$ *to be orthogonal. (10 pts)*

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(3) Compute the projection matrix P onto the orthogonal complement of $\boldsymbol{a}$. *(10 pts)*

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(4) Consider the energy of an arbitrary vector with respect to the matrix P:

$$\begin{bmatrix} x & y & z \end{bmatrix} P \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Compute a formula for the energy and show that it is non-negative for any real numbers x, y, z. *(10 pts)*

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(5) Find x, y, z (not all zero) for which the energy as above is 0. *(5 pts)*

(6) Fill in the blanks. The matrix P is positive ________________________ *(5 pts)*

(7) Fill in the blanks. The vector $\boldsymbol{a}$ and an arbitrary non-zero vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for which the energy is 0 are ____________________ *(5 pts)*