Part 3

Application: Drag

An extended application of dimensional and easy-cases reasoning is these nonlinear, coupled, partial-differential equations:

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}, \quad (3 \text{ eqns})
\]

\[
\nabla \cdot \mathbf{v} = 0. \quad (1 \text{ eqn})
\]

The first three equations, written compactly as one vector equation, are the Navier–Stokes equations of fluid mechanics, and the bottom equation is the continuity equation. The four equations contain the answer to the following questions:
When you drop a paper cone (like a coffee filter) and a smaller cone with the same shape, which has the faster terminal velocity?
Solving those equations is a miserable task, which is why we will instead use our two techniques: dimensions and then extreme cases. For the moment, assume that each cone instantly reaches terminal velocity; that approximation is reasonable but we will check it in ?? using the technique of discretization. So we need to find the terminal velocity. It depends on the weight of the cone and on the drag force $F$ resisting the motion.

### 11.1 Using dimensions

To find the force, we use dimensions and add a twist to handle problems like this one that have an infinity of dimensionally correct answers. The drag force depends on the object’s speed $v$; on the fluid’s density $\rho$; on its kinematic viscosity $\nu$; and on the object’s size $r$. Now
find the dimensions of these quantities and find all dimensionally
correct statements that are possible to make about \( F \). Size \( r \) has dimen-
sions of \( L \). Terminal velocity \( v \) has dimensions of \( LT^{-1} \). Drag force \( F \)
has dimensions of mass times acceleration, or \( MLT^{-2} \). Density \( \rho \) has
dimensions of \( ML^{-3} \). The dimensions of viscosity \( \nu \) are harder. In the
problem set, you show that it has dimensions of \( L^2T^{-1} \). If you look for
combinations of \( v, \rho, \) and \( r, \) and \( v \) that produce dimensions of force,
an infinite number of solutions appear, whereas in previous examples
using dimensions, only one possibility had the correct dimensions.

Hence the need for a more advanced method to handle the infinite
possibilities here. Return to the first principle of dimensions: *you cannot add apples to oranges*. The requirement that the sides of an equation
match dimensionally is one consequence of the apples-and-oranges
principle. Another consequence is that every term in an equation
must have the same dimensions. So imagine any true statement about
drag force:

\[
A + B = C
\]

where \( A, B, \) and \( C \) might be messy combinations of the variables.
Then divide each term by \( A \):

\[
\frac{A}{A} + \frac{B}{A} = \frac{C}{A}.
\]

Because \( A, B, \) and \( C \) have the same dimensions, each ratio is dimen-
sionless. So you can take any (true) statement about drag force and
rewrite it in dimensionless form. No step in this argument depended
on the details of drag. It required only that apples must be added to
apples. So:

You can write any true statement about the world in dimension-
less form.
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Furthermore, you can construct any dimensionless expression using dimensionless groups: products of the variables where the product has no dimensions. Since you can write any true statement in dimensionless form, and can write any dimensionless form using dimensionless groups:

You can write any true statement about the world using dimensionless groups.

In the problem of free fall, with variables $v$, $g$, and $h$, the dimensionless group is $v/\sqrt{gh}$, perhaps raised to a power. With only one group, the only dimensionless statement has the form:

the one group = dimensionless constant,

which results in $v \sim \sqrt{gh}$.

For the drag, what are some dimensionless groups? One group is $F/\rho v^2 r^2$, as you can check by working out its dimensions. A second group is $rv/v$. Any other group, it turns out, can be formed from these two groups. With two groups, the most general dimensionless statement is

one group = $f$(other group),

where $f$ is a dimensionless function. It has a dimensionless argument and must return a dimensionless value because the left side of the equation is dimensionless. Using $F/\rho v^2 r^2$ as the first group:

$$\frac{F}{\rho v^2 r^2} = f \left( \frac{rv}{v} \right).$$

11.2 Extreme cases of the Reynolds number

The second group, which is the quantity in the parentheses, is the Reynolds number and is often written $Re$. It measures how turbulent
the fluid flow is. To find the drag force \( F \), we have to find the function \( f \). It is too hard to determine fully—it would require solving the Navier–Stokes equations—but it might be possible in extreme cases. The extreme cases here are \( Re \to 0 \) and \( Re \to \infty \).

Let’s hope that the falling cones are in one of those limits! To decide, evaluate \( Re \) for the falling cone. From experience, even before you drop the cones to decide which falls faster, either cone falls at roughly \( v \sim 1 \text{ m s}^{-1} \). Its size is roughly \( r \sim 0.1 \text{ m} \). And the viscosity of the fluid (air) in which it falls is \( \nu \sim 10^{-5} \text{ m}^2 \text{ s}^{-1} \), which you can find by looking it up in a table by an online search, or by applying these approximation methods to physics and engineering problems (the theme of another course and book on approximation). So

\[
Re \sim \frac{rv}{\nu} \sim \frac{0.1 \text{ m} \times 1 \text{ m s}^{-1}}{10^{-5} \text{ m}^2 \text{ s}^{-1}} \sim 10^4.
\]

So \( Re \gg 1 \), and we are safe in looking just at that extreme case. Even if the estimate for the speed and size are inaccurate by, say, a factor of 3 each, the Reynolds number is at least 1000, still much larger than 1.

To decide what factors are important in the high-Reynolds-number limit, look at the form of the Reynolds number: \( rv/\nu \). One way to send it to infinity is the limit \( \nu \to 0 \). Viscosity, therefore, becomes irrelevant as \( Re \to \infty \), and in that limit the drag force \( F \) should not depend on viscosity. Although the conclusion is mostly correct, there are subtle lies in the argument. To clarify these subtleties required two hundred years of mathematical and physical development in both theory and experiment. So I will skip the truth, and hope that you are content at least for the moment with almost-truth, especially since it gives the same answer as the truth.

Let’s look at how the requirement of independence from \( \nu \) constrains the general dimensionless form:
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\[
\frac{F}{\rho v^2 r^2} = f(Re)
\]

The left side does not contain viscosity \(v\). The right side might because \(Re\) contains \(v\). So if any Reynolds number shows up on the right side, then viscosity will appear on the right side, with no viscosity on the left side with which to cancel it. And that situation would violate the extreme-case result that, in the \(Re \to \infty\) limit, the drag force is independent of viscosity. So the right side must be independent of \(Re\). Since \(f\) depended only on the Reynolds number, which has just been stricken off the list of allowed dependencies, the right side \(f(Re)\) is a dimensionless constant. Therefore,

\[
\frac{F}{\rho v^2 r^2} = \text{dimensionless constant},
\]

or

\[F \sim \rho v^2 r^2.\]

11.3 Terminal velocity

And now we have the result that we need to find the relative terminal velocity of the large and small cones. The cones reach terminal speed when the drag force balances the weight. The weight is proportional to the area of the paper, so it is proportional to \(r^2\). The drag force is also proportional to \(r^2\), as we just found. To summarize:

\[
\frac{\rho v^2 r^2}{F} \quad \propto \quad \frac{r^2}{\text{weight}}.
\]

The factor of \(r^2\) on each side divides out, so

\[v^2 \propto \frac{1}{\rho},\]

showing that
The cones’ terminal velocity is independent of its size.

That result is indeed what we found in class by doing the experiment. So, without having to solve the Navier–Stokes differential equations, experiment and cheap theory agree!

11.4 What you have learned

When problems become too complicated for the unassisted technique of dimensions, the technique of easy cases complements the technique of dimensions. That symbiosis helped us compute the relative terminal velocities of the falling cones. The general recipe is based on the maxim that You can write any true statement about the world using dimensionless groups. It leads to the following problem-solving plan for finding, say, drag force $F$:

1. Find the quantities on which $F$ depends, and find the dimensions of $F$ and of those quantities.
2. Make dimensionless groups from those quantities.
3. Write the result in general dimensionless form:

   \[
   \text{group containing } F = f(\text{other groups}).
   \]

4. Choosing extreme values for the other dimensionless groups, guess the form of the dimensionless function $f$ in those limits.