Chapter 2
Easy cases

When Einstein was creating his theory of gravitation, he checked that it reproduced the predictions of Newtonian gravity if the speed of light became infinite. When Feynman was creating his path-integral formulation of relativistic quantum mechanics, he checked that it reproduced the results of ordinary, non-relativistic quantum mechanics.

These approaches to discovery are examples of the method of easy cases based on the idea that a correct solution works in all cases – including the easy ones. This tool enables you to check and even guess answers while avoiding hard work. Let's sharpen the tool by trying it in problems from integration, plane geometry, solid geometry, and fluid mechanics.

2.1 Gaussian integral revisited

As the first illustration, retry the Gaussian integral from **Section 1.3**. Here are two possible answers to the integration:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \qquad \sqrt{\pi\alpha} \quad \text{or} \quad \sqrt{\pi/\alpha}?$$

This question was answered in **Section 1.3.3** using dimensions. A second method is to change the integration variable to $z^2 = \alpha x^2$. But forget these two methods temporarily in order to practice easy cases.

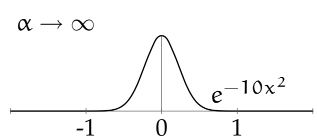
The correct solution works for all α, or for all α where the integral makes sense. That caveat excludes the range $\alpha < 0$ and maybe even $\alpha = 0$, so the solution must work in the remaining range $\alpha > 0$. In that range, choose easy cases: values of α where the integral is easy to evaluate.

2.1.1 Extreme values of α

The integral is easy in the particular cases $\alpha = \infty$ or $\alpha = 0$: the two extremes of the range $\alpha \geqslant 0$. This observation is the basis of a rule of thumb:

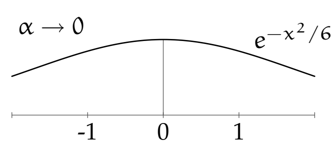
> Extreme cases are often easy cases.

▶ *What happens when $\alpha = \infty$?*



As the first easy case, try the extreme $\alpha = \infty$ or rather approach the extreme by imagining that α rises to ∞. Then the exponent $-\alpha x^2$ becomes very negative even when x is close to zero. The exponential of a large negative number is nearly zero, so the bell curve narrows to a sliver, and its area shrinks toward zero. Therefore, as $\alpha \to \infty$ the integral shrinks to zero. This result refutes the first choice $\sqrt{\pi\alpha}$, which goes to infinity as $\alpha \to \infty$. The second choice $\sqrt{\pi/\alpha}$ correctly goes to zero, so it passes the $\alpha = \infty$ test.

▶ *What happens when $\alpha = 0$?*



The other extreme $\alpha = 0$ provides the second easy case. Near that extreme, the bell curve flattens into a horizontal line with height 1. Integrated over an infinite range, its area is infinite. This result refutes the first choice $\sqrt{\pi\alpha}$, is zero when $\alpha = 0$. The second choice $\sqrt{\pi/\alpha}$ correctly goes to infinity, so it passes the $\alpha = 0$ test.

In short, the second option passes both easy-case tests and the first option fails them. If the two options were the only choices, then choose $\sqrt{\pi/\alpha}$. Suppose, however, that $\sqrt{2/\alpha}$ joins the choices. How could you decide between $\sqrt{2/\alpha}$ and $\sqrt{\pi/\alpha}$? Both behave correctly in the two extreme cases. They even have identical dimensions, so dimensions are also not decisive.

2.1.2 The almost-as-easy case $\alpha = 1$

To make the choice, try a third easy case that lies between the extremes $\alpha = 0$ and $\alpha = \infty$: the almost-as-easy case $\alpha = 1$. In that case the integral simplifies to

$$I \equiv \int_{-\infty}^{\infty} e^{-x^2}\, dx,$$

where the $\equiv$ notation means 'is defined to be' rather than the more common mathematics meaning of modulo. This integral is the simplest Gaussian integral and, as demonstrated in this section, its value is $\sqrt{\pi}$. The method of evaluation is to construct I^2 and to rewrite that integral in polar coordinates, whereupon it becomes simple.

2.1.2.1 Constructing I^2

Here is I^2:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right) \times \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right).$$

The integration variables are arbitrary. So, in the second factor, choose y as the integration variable:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right) \times \left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right).$$

This choice gives the integration range a simple geometric interpretation. To see it, group the integral signs together and the integrands e^{-x^2} and e^{-y^2} together:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dx\, dy.$$

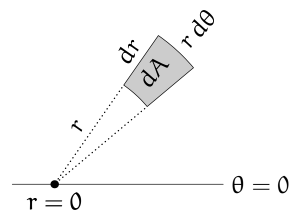
The two exponentials multiply together to make $e^{-(x^2+y^2)}$. Since $x^2+y^2 = r^2$ where r is distance from the origin, the integrand is e^{-r^2}. This function is integrated over all possible x and y: in other words, over the entire plane. So I^2 is the integral of e^{-r^2} over the whole plane:

$$I^2 = \int_{\text{plane}} e^{-r^2}\, dA,$$

where dA is the area element.

2.1.2.2 Evaluating in polar coordinates

An integrand with r as its only variable suggests polar coordinates. In polar coordinates, the area element is $dA = r\,dr\,d\theta$, so



$$I^2 = \int_{\text{plane}} e^{-r^2} \underbrace{r\,dr\,d\theta}_{dA}.$$

The integration region – the entire plane – contains all possible r and all possible θ, so r goes from 0 to ∞, and θ goes from 0 to 2π:

$$\int_{\text{plane}} dA = \int_0^{2\pi}\int_0^{\infty} r\,dr\,d\theta$$

So I^2 is

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} \underbrace{r\,dr\,d\theta}_{dA}.$$

Now shuffle the pieces and integral signs to separate the θ and r integrals:

$$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r\,dr.$$

The θ integral is simply 2π. The r integrand re^{-r^2} is easy to integrate because of the factor of r, which is almost the derivative of the r^2 in the exponent:

$$\int e^{-r^2} r\,dr = -\frac{1}{2}e^{-r^2} + \text{constant},$$

and

$$\int_0^{\infty} e^{-r^2} r\,dr = \frac{1}{2}.$$

Now multiply the r integral and the θ integral:

$$I^2 = \underbrace{\theta \text{ integral}}_{2\pi} \times \underbrace{r \text{ integral}}_{1/2} = \pi$$

The Gaussian integral I is the square root of I^2:

$$I \equiv \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Although the easy case $\alpha = 1$ was not as easy as $\alpha = 0$ or $\alpha = \infty$, setting $\alpha = 1$ makes α vanish from the calculation, thereby simplifying it.

2.1.3 Using the easy-case tests

The third easy-case test $\alpha = 1$ helps pass judgment on various choices (guesses) for the original integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx.$$

Among the choices, $\sqrt{2/\alpha}$, $\sqrt{\pi/\alpha}$, and $\sqrt{\pi\alpha}$, only $\sqrt{\pi/\alpha}$ passes all three tests, so

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}.$$

In summary, check any formula using easy cases:

1. Pick parameter values that make the analysis easy. These values are the easy cases.
2. In the easy cases, predict the behavior without doing hard work.
3. Check that the predictions match the proposed formula. If not, throw it out and ask for or invent another!

Easy cases are not the only way to check formulas. Dimensions also can narrow the Gaussian-integral choices, as was shown in **Section 1.3**. Furthermore, dimensions refute many choices that pass the easy-case tests – for example $\sqrt{\pi}/\alpha$ and $\sqrt{\pi}/\alpha^2$.

However, easy cases are still useful. First, they are quick. They do not require inventing or computing dimensions for x, α, dx, and the whole integral – the extensive analysis of **Section 1.3**. Second, easy cases can decide between choices with identical dimensions like $\sqrt{2/\alpha}$ and $\sqrt{\pi/\alpha}$. So, keep both tools ready in your toolbox.

Problem 2.1 Which easy-case tests does it pass?
Consider the proposal

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \frac{\sqrt{\pi}-1}{\alpha}+1.$$

Which easy-case tests does it pass?

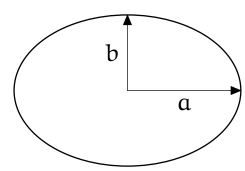
Problem 2.2 Passing the three tests
Check that $\sqrt{\pi}/\alpha$ and $\sqrt{\pi}/\alpha^2$ pass the three easy-case tests $\alpha = 0$, $\alpha = 1$, and $\alpha = \infty$.

Problem 2.3 Plausible, incorrect alternative
Can you invent an alternative to $\sqrt{\pi/\alpha}$ that has valid dimensions and passes all three easy-case tests?

2.2 Area of an ellipse

Easy cases work not only in calculus. They are useful in any problem where a formula needs to be checked or guessed – for example, areas and volumes of geometric shapes.

This ellipse has semimajor axis a and semiminor axis b, and here are several candidates for its area A:



a. ab^2

b. $a^2 + b^2$

c. a^3/b

d. $2ab$

e. πab

What are the merits, or otherwise, of each choice?

Let's examine each candidate using easy cases.

$A = ab^2$. This candidate has dimensions of L^3, whereas area has dimensions of L^2. So it fails the dimensions test and does not even graduate to the easy-cases tests. However, the remaining choices have correct dimensions, so they require the method of easy cases.

$A = a^2 + b^2$. Likely easy cases are the extreme values of the parameters a and b. So try $a = 0$ to produce an infinitesimally thin ellipse. This ellipse has zero area, yet when $a = 0$ the candidate $A = a^2 + b^2$ predicts $A = b^2$. So the candidate fails the $a = 0$ easy-case test.

$A = a^3/b$. When $a = 0$, this candidate correctly predicts zero area. Since $a = 0$ was a useful test, and the axis labels a and b are almost interchangeable, $b = 0$ should also be a useful easy case. It also produces an infinitesimally thin ellipse with zero area; alas, the candidate a^3/b predicts infinite area, so it fails the new $b = 0$ test.

$A = 2ab$. This candidate shows promise: When $a = 0$ or $b = 0$, the actual and predicted areas are zero, so it passes both easy-case tests. Another easy case for a and b is when $a/b = 1$ or when $a = b$. Then the ellipse becomes a circle with radius a and area πa^2. The candidate $2ab$, however, predicts $A = 2a^2$ (or $A = 2b^2$), so it fails the new $a = b$ test.

$A = \pi ab$. This candidate passes all three tests: $a = 0$, $b = 0$, and $a = b$. With every test that a candidate passes, our confidence in it increases. Indeed, πab is the correct area (**Problem 2.4**).

Problem 2.4 Area by calculus
Show using integration that $A = \pi ab$. See **Problem 7.1** for an alternative method.

Problem 2.5 Generalization
What is the volume of an ellipsoid with principal radii a, b, and c?

Problem 2.6 Inventing a passing candidate
Can you invent a second candidate for the area that has correct dimensions and passes the $a = 0$, $b = 0$, and $a = b$ tests?

2.2.1 How to choose extreme cases

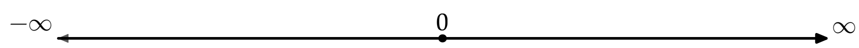
In analyzing the candidates for the area of an ellipse, one easy case was the extreme $a = 0$, and another easy case was its symmetric counterpart $b = 0$. Less obviously, symmetry also suggests the third easy case $a = b$.

To see how, notice that the symmetry between a and b requires that they have identical dimensions. Therefore a natural comparison is their dimensionless ratio a/b. This ratio ranges between 0 and ∞:

And this range has two obviously special points: the endpoints 0 and ∞ corresponding to the first two easy cases $a = 0$ and $b = 0$ respectively.

However, the ratio a/b breaks the full symmetry between a and b. For example, the condition $0 \leqslant a \leqslant b$ restricts the ratio to a finite range, $[0, 1]$; whereas the symmetric condition $0 \leqslant b \leqslant a$ restricts the ratio to an infinite range, $[1, \infty]$.

Fixing this broken symmetry will make $a = b$ a natural easy case. One fully symmetric, dimensionless combination of a and b is $\log(a/b)$. This combination ranges between $-\infty$ and ∞:

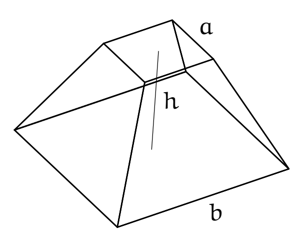


The special points in this range include the endpoints $-\infty$ and ∞ corresponding again to the easy cases $a = 0$ and $b = 0$ respectively; and the midpoint 0 corresponding to the third easy case $a = b$. In short, extreme cases are not the only easy cases; and easy cases also arise by equating symmetric quantities.

Problem 2.7 Other symmetric combinations
Invent other symmetric, dimensionless combinations of a and b – such as $(a - b)/(a + b)$. Investigate whether those combinations have $a = b$ as an interesting point.

2.3 Volume of a truncated pyramid

The two preceding examples – the Gaussian integral (**Section 2.1**) and the area of an ellipse (**Section 2.2**) – used easy cases to check proposed formulas, as a method of analysis. The next level of sophistication – the next level in Bloom's taxonomy [6] – is to use easy cases as a method of synthesis.



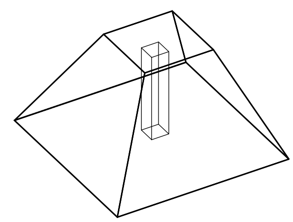
As an example, start with a pyramid with a square base and slice a piece from its top using a knife parallel to the base. This so-called frustum has a square base and square top parallel to the base. Let its height be h, the side length of the base be b, and the side length of the top be a. What is its volume?

Since volume has dimensions of L^3, candidates such as ab, ab^3, and bh are impossible. However, dimensions cannot distinguish among choices with correct dimensions, such as ab^2, abh, or even a^2b^2/h. Further progress requires creating easy-cases tests.

2.3.1 Easy cases

The simplest test is $h = 0$: a pyramid with zero height and therefore zero volume. This test eliminates the candidate a^2b^2/h, which has correct dimensions but, when $h = 0$, incorrectly predicts infinite volume.

▶ *How should volume depend on height?*



Candidates that predict zero volume include products such as ha^2 or h^2a. To choose from such candidates, decide how the volume V should depend on h, the height of the solid. This decision is aided by a thought experiment. Chop the solid into vertical slivers, each like an oil-drilling core (see figure), then vary h. For example, doubling h doubles the height and volume of each sliver; therefore doubling h doubles V. The same thought experiment shows that tripling h triples V. In short: $V \propto h$, which rules out h^2a as a possible volume.

Problem 2.8 Another reason that $V \propto h$
Use the easy case of $a = b$ to argue that $V \propto h$.

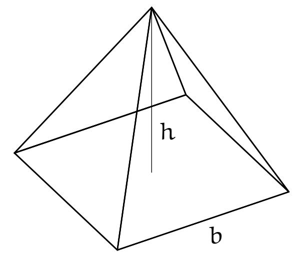
The constraint $V \propto h$, together with the requirement of dimensional correctness, means that

$$V = h \times \text{function of } a \text{ and } b \text{ with dimensions of } L^2.$$

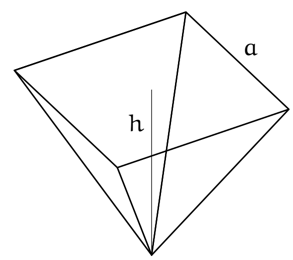
Further easy-cases tests help synthesize that function.

▶ *What are other easy cases?*

A second easy case is the extreme case $a = 0$, where the top surface shrinks to a point. The symmetry between a and b suggests the extreme case $b = 0$ as another easy case. The symmetry also suggests $a = b$ as a non-extreme easy case. Let's apply the three new tests in turn, developing formulas to synthesize a candidate that passes all the tests.



$a = 0$. In this extreme case, the truncated pyramid becomes an ordinary pyramid with height h and square base of side length b. So that its volume has dimensions of L^3 and is proportional to h, its volume must have the form $V \sim hb^2$. This form, which does not specify the dimensionless constant, stands for a family of candidates. Perhaps this family contains the correct volume for the truncated pyramid? Each family member passes the $a = 0$ test, by construction. How do they fare with the other easy cases?



$b = 0$. In this extreme case, the truncated pyramid becomes an upside-down but otherwise ordinary pyramid. All the candidates $V \sim hb^2$ predict zero volume when $b = 0$, so all fail the $b = 0$ test. The symmetric alternatives $V \sim ha^2$ pass the $b = 0$ test; unfortunately, they fail the $a = 0$ test. Are we stuck?

▶ *Invent a candidate that passes the $a = 0$ and $b = 0$ tests.*

To a family of candidates that pass the $a = 0$ and $b = 0$ tests, add the two families that pass each test to get $V \sim ha^2 + hb^2$ or

$$V \sim h(a^2 + b^2).$$

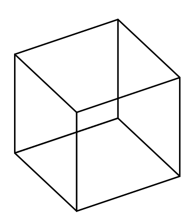
Two other families of candidates that pass both tests include

$$V \sim h(a + b)^2.$$

and

$$V \sim h(a - b)^2.$$

Choosing among them requires the last easy case: $a = b$.



$a = b$. When $a = b$ – the easiest of the last three cases – the truncated pyramid becomes a rectangular prism with height h, base area b^2 (or a^2), and volume hb^2. When $a = b$, the family of candidates $V \sim h(a^2 + b^2)$ predicts hb^2 when the dimensionless constant is $1/2$. So

$$V = \frac{1}{2}h(a^2 + b^2)$$

passes the $a = b$ test. When $a = b$, the family of candidates $V \sim h(a+b)^2$ predicts the correct volume when the dimensionless constant is $1/4$. So

$$V = \frac{1}{4}h(a+b)^2$$

passes the $a = b$ test. However, the family of candidate $V \sim h(a-b)^2$ predict zero volume when $a = b$, so they all fail the $a = b$ test.

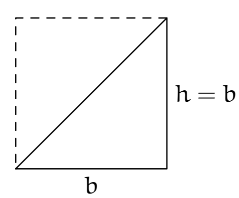
To decide between the survivors $V = h(a+b)^2/2$ and $V = h(a+b)^2/4$, return to the easy case $a = 0$. In that extreme case, the survivors predict $V = hb^2/2$ and $V = hb^2/4$, respectively. These predictions differ in the dimensionless constant. If we could somehow guess the correct dimensionless constant in the volume of an ordinary pyramid, we can decide between the surviving candidates for the volume of the truncated pyramid.

▶ *What is the dimensionless constant?*

2.3.2 Finding the dimensionless constant

Finding the dimensionless constant looks like a calculus problem: Slice an ordinary pyramid into thin horizontal sections, then add (integrate) their volumes. A simple but surprising alternative is the method of easy cases – surprising because easy cases only rarely determine a dimensionless constant.

The method is best created with an analogy: Rather than guessing the dimensionless constant in the volume of a pyramid, solve a similar but simpler problem – a method discussed in detail in **Chapter 6**. In this case, let's invent a two-dimensional shape and find the dimensionless constant in its area.

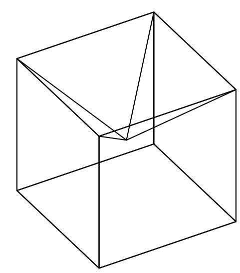


An analogous shape is a triangle with base b and height h. What is the dimensionless constant in its $A \sim bh$? To reap the full benefit of the analogy, answer that question using easy cases rather than calculus. So, choose b and h to make an easy triangle. The easiest triangle is a 45° right triangle with $h = b$. Two of these triangles form an easy shape – a square with area b^2 – so the area of one triangle is $A = b^2/2$ when $h = b$. Therefore, the dimensionless constant is $1/2$ and $A = bh/2$.

Now extend this reasoning to our three-dimensional solid: What square-based pyramid, combined with itself a few times, makes an easy solid? Choosing a pyramid means choosing its base length b and its height h.

However, only the aspect ratio h/b matters in the analysis – as in the triangle example. So, our procedure will be to choose a goal solid and then to choose a convenient b and h so that such pyramids combine to form the goal shape.

▶ *What is a convenient goal solid?*

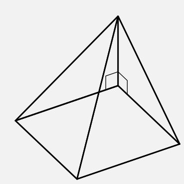


A convenient goal solid is suggested by the square pyramid base – perhaps one face of a cube? If so, the cube would be formed from six pyramids. To choose their b and h, imagine how the pyramids fit into a cube. With the base of each pyramid forming one face of the cube, the tips of the pyramids point inward and meet in the center of the cube. To make the tips meet, the pyramid's height must be $b/2$, and six of those pyramids make a cube of side length b and volume b^3.

To keep b and h integers, choose $b = 2$ and $h = 1$. Then six pyramids make a cube with volume 8, and the volume of one pyramid is 4/3. Easy cases have now excavated sufficient information to determine the dimensionless constant: Since the volume of the pyramid is $V \sim hb^2$ and since $hb^2 = 4$ for these pyramids, the missing constant must be 1/3. Therefore

$$V = \frac{1}{3}hb^2.$$

Problem 2.9 Vertex location



The six pyramids do not make a cube unless each pyramid's top vertex is directly above the center of the base. So the result $V = hb^2/3$ might apply only in that special case. If instead the top vertex is above one of the base vertices, what is the volume?

Problem 2.10 Triangular base

Guess the volume of a pyramid with a triangular base.

2.3.3 Using the magic factor of one-third

The purpose of the preceding easy-cases analysis for an ordinary pyramid was to decide between two candidates for the volume of a truncated pyramid: $V = h(a^2 + b^2)/2$ and $V = h(a + b)^2/4$. Unfortunately, neither candidate predicts the correct volume $V = hb^2/3$ for an ordinary pyramid ($a = 0$). Oh, no!

We need new candidates. One way to generate them is first to rewrite the two families of candidates that passed the $a = 0$ and $b = 0$ tests:

$$V \sim a^2 + b^2 = a^2 + b^2,$$
$$V \sim (a+b)^2 = a^2 + 2ab + b^2,$$
$$V \sim (a-b)^2 = a^2 - 2ab + b^2.$$

The expanded versions on the right have identical a^2 and b^2 terms but differ in the ab term. This variation suggests an idea: that by choosing the coefficient of ab, the volume might pass all easy-cases tests. Hence the following three-part divide-and-conquer procedure:

1. Choose the coefficient of a^2 to pass the $b = 0$ test.
2. Choose the coefficient of b^2 to pass the $a = 0$ test. Choosing this coefficient will not prejudice the already passed $b = 0$ test, because when $b = 0$ the b^2 term vanishes
3. Finally, choose the coefficient of ab to pass the $a = b$ test. Choosing this coefficient will not prejudice the already passed $b = 0$ and $a = 0$ tests, because in either case ab vanishes.

The result is a volume that passes the three easy-cases tests: $a = 0$, $b = 0$, and $a = b$.

To pass the $b = 0$ test, the coefficient of a^2 must be $1/3$ – the result of combining six pyramids into a cube. Similarly, to pass the $a = 0$ test, the coefficient of b^2 must also be $1/3$. The resulting family of candidates is:

$$V = \frac{1}{3}h(a^2 + nab + b^2).$$

Among this family, one must pass the $a = b$ test. When $a = b$, the candidates predict

$$V = \frac{2+n}{3}hb^2.$$

When $a = b$, the truncated pyramid becomes a rectangular prism with volume hb^2, so the coefficient $(2+n)/3$ should be 1. Therefore $n = 1$, and the volume of the truncated pyramid is

$$V = \frac{1}{3}h(a^2 + ab + b^2).$$