candidate predicts the correct volume $V = hb^2/3$ for an ordinary pyramid ($a = 0$). Oh, no!

We need new candidates. One way to generate them is first to rewrite the two families of candidates that passed the $a = 0$ and $b = 0$ tests:

$$
\begin{aligned}
V &\sim a^2 + b^2 = a^2 + b^2,\\
V &\sim (a+b)^2 = a^2 + 2ab + b^2,\\
V &\sim (a-b)^2 = a^2 - 2ab + b^2.
\end{aligned}
$$

The expanded versions on the right have identical a^2 and b^2 terms but differ in the ab term. This variation suggests an idea: that by choosing the coefficient of ab, the volume might pass all easy-cases tests. Hence the following three-part divide-and-conquer procedure:

1. Choose the coefficient of a^2 to pass the $b = 0$ test.
2. Choose the coefficient of b^2 to pass the $a = 0$ test. Choosing this coefficient will not prejudice the already passed $b = 0$ test, because when $b = 0$ the b^2 term vanishes
3. Finally, choose the coefficient of ab to pass the $a = b$ test. Choosing this coefficient will not prejudice the already passed $b = 0$ and $a = 0$ tests, because in either case ab vanishes.

The result is a volume that passes the three easy-cases tests: $a = 0$, $b = 0$, and $a = b$.

To pass the $b = 0$ test, the coefficient of a^2 must be $1/3$ – the result of combining six pyramids into a cube. Similarly, to pass the $a = 0$ test, the coefficient of b^2 must also be $1/3$. The resulting family of candidates is:

$$V = \frac{1}{3}h(a^2 + nab + b^2).$$

Among this family, one must pass the $a = b$ test. When $a = b$, the candidates predict

$$V = \frac{2+n}{3}hb^2.$$

When $a = b$, the truncated pyramid becomes a rectangular prism with volume hb^2, so the coefficient $(2+n)/3$ should be 1. Therefore $n = 1$, and the volume of the truncated pyramid is

$$V = \frac{1}{3}h(a^2 + ab + b^2).$$

Problem 2.11 Integration
Use integration to check that $V = h(a^2 + ab + b^2)/3$.

Problem 2.12 Triangular pyramid
Instead of a pyramid with a square base, start with a pyramid with an equilateral triangle of side length b as its base. Then make the usual truncated pyramid by slicing a piece off the top using a plane parallel to the base. If the top is an equilateral triangle of side length a, and the height is h, what is the volume of this truncated pyramid? (See also **Problem 2.10**.)

Problem 2.13 Truncated cone
What is the volume of a truncated cone with a circular base of radius r_1 and circular top of radius r_2 (with the top parallel to the base)?

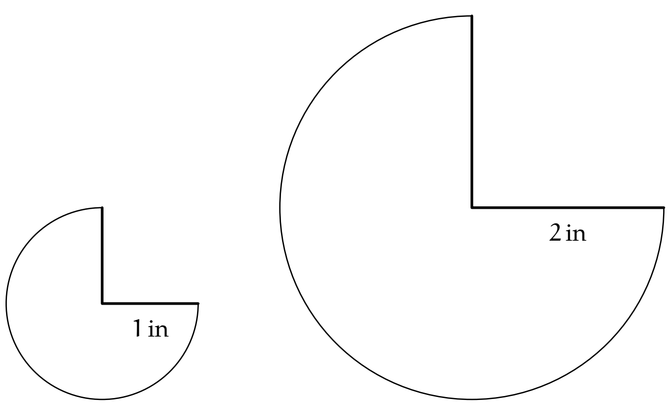
2.4 Drag: Using dimensions and easy cases

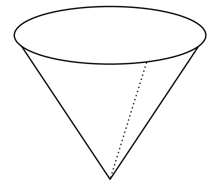
Equations often exceed the capabilities of known mathematics, whereupon easy cases and other street-fighting tools are among the few ways to progress. As an example, consider the frightening Navier–Stokes equations of fluid mechanics:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu\boldsymbol{\nabla}^2\mathbf{v},$$

where $\mathbf{v}$ is the velocity of the fluid (as a function of position and time), ρ is density, p is pressure, and ν is kinematic viscosity. These equations explain an amazing variety of phenomena: turbulence, supersonic flight, energy cost of bicycling, river rapids, and so much else in the world.

Here is a home experiment showing one such phenomena. Photocopy this page, magnifying it by a factor of 2, and cut out these templates:





Tape together the thickly drawn edges to make one cone from each template. The cones have the same shape with the large cone having double the linear dimensions (height and width) of the small cone.

When the cones are dropped point downward, what is the approximate ratio of their terminal velocities?

To use the Navier–Stokes equations to predict the terminal velocity of a cone:

1. Impose boundary conditions, which include the motion of the cone and the requirement that no fluid (air) enters its paper;
2. Solve the equations together with the continuity equation $\nabla\cdot\mathbf{v} = 0$ to find the pressure and velocity gradient at the surface of the cone;
3. Use the pressure and velocity gradient to find the net force and torque on the cone;
4. Use the net force and torque to find the motion of the cone, ensuring that the motion is consistent with the motion assumed in **Step 1**. This step is very difficult!

Alas, the Navier–Stokes equation are coupled, nonlinear partial-differential equations with probably no general solution. Solutions are known only in particular cases with simple geometry: for example, a sphere moving very slowly in a viscous fluid, or a sphere moving at any speed in a zero-viscosity fluid. This solution, valid at any speed, seems general on first glance. However, a zero-viscosity fluid – what Feynman calls 'dry water' [7, Vol. I, Chap. 40] – is a fiction. It would not produce any drag: Drag arises solely from the dissipation produced by viscosity. Without viscosity, therefore, the cones in the home experiment would fall freely like a rock or as if in a vacuum – contrary to everyday experience. So the zero-viscosity solution cannot help us analyze the falling cones.

Even if our mathematics can someday include viscosity when solving the high-speed flow around a sphere, there is little hope for solving the complicated flow around an irregular, quivering shape such as a flexible cone made of paper. These difficulties encourage us to find an alternative approach.

Problem 2.14 Checking dimensions in the Navier–Stokes equations
Check that the first three terms of the Navier–Stokes equations have identical dimensions.

Problem 2.15 Dimensions of kinematic viscosity
Use the Navier–Stokes equations to deduce the dimensions of kinematic viscosity ν.

2.4.1 Using dimensions

Let's try our two tools of dimensions and easy cases. A direct approach is to use these tools to deduce the terminal velocity itself. An indirect approach is to deduce the drag force as a function of fall speed and then to find the speed at which the drag balances the weight of the cones. This two-step approach introduces one new quantity, the drag force, while eliminating two quantities: the gravitational acceleration and the mass of a cone. It thereby simplifies the hard step of applying dimensions to a problem with many quantities.

Problem 2.16 Explaining the simplification
Why the drag force independent of the gravitational acceleration g and the cone's mass m (whereas it depends on the shape and size)?

The principle of dimensions is that every term in a valid equation has identical dimensions. Applied to the drag force F, it means that in the equation

$$F = f(\text{quantities that affect } F),$$

both sides must have dimensions of force. Let's find the quantities that affect F, then combine them to make a quantity with dimensions of force.

▶ *On what quantities does the drag depend?*

The drag force depends on:

1. v, the speed of the cone;
2. ρ, the density of the fluid;
3. ν, the kinematic viscosity of the fluid;

4. r, the characteristic size of the cone; and
5. the shape of the cone.

What are the dimensions of these quantities?

To find out how to combine quantities into a force, the first step is to list their dimensions. Shape – whether an object is square, oblate, etc. – is a dimensionless concept, so it can appear arbitrarily in the equation

$$F = f(\text{quantities that affect } F).$$

In other words, dimensions cannot determine how shape affects the drag force F. So, ignore the effect of shape.

The other quantities have the following dimensions. Size r has dimensions of L. Speed v has dimensions of LT^{-1}. Density ρ has dimensions of ML^{-3}. And, as you show in **Problem 2.15**, viscosity ν has dimensions of L^2T^{-1}.

What combinations, if any, of v, ρ, ν, and r have dimensions of force?

The next step is to find all combinations of v, ρ, ν, and r that have dimensions of force. In earlier dimensions examples – free fall (**Section 1.2**) and integration (**Section 1.3**) – only one combination of the relevant quantities had the correct dimensions. However, in this drag-force problem, the four quantities v, ρ, ν, and r produce an infinity of combinations with dimensions of force. Here are examples:

$$F_1 = \rho v^2 r^2,$$
$$F_2 = \rho \nu v r,$$

or meta-combinations such $\sqrt{F_1 F_2}$ or F_1^2/F_2. Any sum of these meta-combinations is also a force, so the drag force F might be

$$\sqrt{F_1 F_2} + \frac{F_1^2}{F_2}$$

or

$$F_1^{2/3} F_2^{1/3} - \frac{F_1^3}{F_2^2}$$

How to decide among these possibilities? We need a method more sophisticated than the hunt-and-peck method of guessing combinations with correct dimensions.

To develop the sophisticated approach, return to the first principle of dimensions – that you cannot add apples to oranges or, equivalently, that all terms in an equation have identical dimensions. This principle applies to all valid equations, including any statement about drag, such as

$$A + B = C$$

where the blobs labeled A, B, and C contain the quantities F, v, ρ, ν, and r.

Although the blobs A, B, and C might be absurdly complex functions, they have identical dimensions. Therefore, make a dimensionless equation by dividing each term by A:

$$\frac{A}{A} + \frac{B}{A} = \frac{C}{A}.$$

Therefore, the statement $A + B = C$ can be written in dimensionless form. The same method of rewriting works for any statement about drag, such as $A = B$ or $A + B - C = D$, because all terms must have identical dimensions. Hence any statement about drag force can be rewritten in dimensionless form. For the same reason, this method of rewriting is not limited to statements about drag. In other words:

> Any (true) statement about the world can be written in dimensionless form.

Any dimensionless expression is composed of dimensionless products of the variables – called dimensionless groups. Since any true statement can be written in dimensionless form, and any dimensionless form can be written using using dimensionless groups:

> All (true) statements about the world can be constructed from dimensionless groups.

▶ *Is the free-fall example (**Section 1.2**) consistent with this principle?*

Before applying this principle to the complicated problem of drag, try it in the simple example of free fall (**Section 1.2**). The exact impact speed of an object dropped from a height h is $v = \sqrt{2gh}$, where g is gravitational acceleration. This result can indeed be written in the dimensionless form $v/\sqrt{gh} = \sqrt{2}$, and this form uses only the dimensionless group $v/\sqrt{gh}$. Our new principle passes its first test.

This dimensionless-group analysis of formulas, when reversed, becomes a method of synthesis: a method for guessing formulas. Let's warm up by synthesizing the impact speed v for the free-fall example. First, list the quantities in the problem, which here are v, g, and h. Second, combine these quantities into dimensionless groups. Many groups are possible: for example $v/\sqrt{gh}$, v^2/gh, or $v^4/(gh)^2$. Third, bring order to these possibilities by looking for the smallest set of sufficient groups. In this case, all groups can be constructed from one group. The choice of that group is not constrained by mathematics; however, some choices have simple physical interpretations. The group v^2/gh is also mv^2/mgh, where m is the mass of the falling object. That ratio is, except for a factor of 2, a kinetic energy divided by a potential energy. So v^2/gh measures the relative importance of kinetic and potential energies. Enchanted by this physical interpretation, let's choose v^2/gh, rather than $v/\sqrt{gh}$, as the base dimensionless group.

The fourth step, which is where the magic happens, is to construct possible dimensionless statements. With only one group, the only possible dimensionless statement is

$$\frac{v^2}{gh} = \text{dimensionless constant.}$$

In other words, $v^2/gh \sim 1$ or $v \sim \sqrt{gh}$. Using dimensionless groups, we've synthesized a formula for the impact speed.

What's the use of dimensionless groups?

The hunt-and-pack method of dimensions in **Section 1.2** produced the same formula. So, why bother with the complications of finding dimensionless groups? They do not help much in simple problems with only one dimensionless group, when any method akin to dimensions will easily synthesize the only possible formula:

$$\text{dimensionless group} \sim 1.$$

However, in problems with more than one dimensionless group – such as the drag force – the dimensionless-groups method is essential.

Problem 2.17 Fall time
Synthesize an approximate formula for the fall time (rather than the impact speed) from the quantities t (fall time), g, and h.

Problem 2.18 Kepler's third law
Synthesize Kepler's third law that connects orbital period to orbital radius. (See also **Problem 1.25**.)

▶ *What dimensionless groups can be constructed for the drag problem?*

For the drag problem, form dimensionless groups from among the goal quantity F and the quantities on which F depends: v, ρ, ν, and r. One group is $F/\rho v^2 r^2$, as you can check by working out its (lack of) dimensions. A second group is rv/ν. All groups, it turns out, can be formed from these two groups (**Problem 2.19**). With two independent groups, the most general dimensionless statement is

$$\text{one group} = f(\text{other group}),$$

where the function f produces a dimensionless value, to match the dimensionless left side.

▶ *Which of these two groups belongs on the left side?*

The goal is to synthesize a formula for F, and F appears only in the group $F/\rho v^2 r^2$. So, place that group on the left side rather than wrapping it in the still-mysterious function f. With this choice, the general form for statements about drag force is

$$\frac{F}{\rho v^2 r^2} = f\left(\frac{rv}{\nu}\right).$$

Problem 2.19 Only two groups
Show that F, v, ρ, ν, and r produce only two independent dimensionless groups.

Problem 2.20 How many groups in general?
Is there a general way to determine the number of independent dimensionless groups? (The answer was given in 1914 by Buckingham [8].)

This sophisticated application of dimensions used the following procedure:

1. Find the quantities on which F depends.
2. Construct independent dimensionless groups from F and those quantities.
3. Write the general statement about F in dimensionless form:

$$\text{group containing } F = f(\text{other groups}).$$

The procedure might seem pointless, not having produced a drag force except as a formula with an unknown function f. On the one hand, the procedure has greatly improved our chances of finding f. The original problem required guessing the four-variable function h in $F = h(v, \rho, \nu, r)$. Using dimensions simplifies the problem to guessing a function of only one variable:

$$\frac{F}{\rho v^2 r^2} = f\left(\frac{rv}{\nu}\right).$$

The magnitude of this simplification is described by the Cambridge statistician and physicist Harold Jeffreys [9, p. 82]:

> A good table of functions of one variable may require a page; that of a function of two variables a volume; that of a function of three variables a bookcase; and that of a function of four variables a library.

Problem 2.21 Dimensionless groups for the truncated pyramid
The truncated pyramid of **Section 2.3** has volume

$$V = \frac{1}{3}h(a^2 + ab + b^2).$$

Make dimensionless groups from V, h, a, and b, and rewrite this formula using the groups. [There are many choices and ways to do so.]

2.4.2 Using easy cases

Greatly improved though our chances may be, they do not look high: Even the simpler, one-variable drag problem has no exact solution. But, it might have exact solutions in particular cases. The likely places to look are the easy cases, the the easiest cases are often extreme cases, so first try extreme cases.

▶ *Extreme cases of what?*

To decide what will take extreme values, look at the general form:

$$\frac{F}{\rho v^2 r^2} = f\left(\frac{rv}{\nu}\right).$$

The changeable quantity – the input – is rv/ν, so try its extremes. Before doing so, first figure out the meaning of rv/ν – otherwise one easily lapses into a mindless game of symbol pushing.

What is the physical interpretation of rv/ν?

This famous combination rv/ν is the Reynolds number – often denoted Re. To uncover its physical interpretation, look at the terms of the Navier–Stokes equations and compare the magnitude of the inertial term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ with the magnitude of the viscosity term $\nu\nabla^2\mathbf{v}$.

What is the approximate magnitude of $(\mathbf{v}\cdot\nabla)\mathbf{v}$?

The inertial term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ has one spatial derivative ∇ and two velocities $\mathbf{v}$. So, using the significant-change approximation (**Section 1.4.3**):

$$(\mathbf{v}\cdot\nabla)\mathbf{v} \sim \frac{(\text{significant change in flow velocity})^2}{\text{distance over which flow velocity changes significantly}}.$$

First estimate the numerator. The fluid motion is produced by the cone's motion, falling at speed v. Therefore flow velocities near the cone are comparable to v (and, in a few spots, somewhat larger than v). Whereas far from the cone the air hardly moves. So v, or a similar value, is a typical flow velocity and also a significant change in flow velocity.

For the denominator, estimate the distance over which the flow velocity changes significantly, for example to $v/2$. As mentioned, far from the cone the air hardly moves. How far counts as 'far from the cone'? It means far compared to the cone's size r, because it is the size of the flow disturbance. So in a distance r, the flow velocity changes significantly.

Therefore the inertial term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ is roughly

$$(\mathbf{v}\cdot\nabla)\mathbf{v} \sim \frac{v^2}{r}.$$

What is the approximate magnitude of the viscosity term $\nu\nabla^2\mathbf{v}$?

The viscosity term $\nu\nabla^2\mathbf{v}$ contains two spatial derivatives, one flow velocity, and one viscosity. The typical change in flow velocity is v – as in the inertial term. Each spatial derivative again contributes a factor of $1/r$ to the typical magnitude. So viscosity term $\nu\nabla^2\mathbf{v}$ is roughly

$$\nu\nabla^2\mathbf{v} \sim \frac{\nu v}{r^2}.$$

Here is ratio of the inertial term to the viscous term:

$$\frac{(\mathbf{v}\cdot\nabla)\mathbf{v}}{\nu\nabla^2\mathbf{v}} \sim \frac{v^2/r}{\nu v/r^2} = \frac{rv}{\nu}.$$

Not only is the ratio dimensionless (why must it be?), but it is the Reynolds number! So the Reynolds number measures how much viscosity affects the flow. When $\mathrm{Re} \ll 1$, the viscous term is the far larger term, and viscosity is the important physical effect. The flow oozes, as when pouring cold ketchup. When $\mathrm{Re} \gg 1$, the viscous term is the far smaller term, and viscosity is a negligible effect. It then cannot prevent nearby pieces of fluid from acquiring significantly different velocities, so the flow easily turns turbulent.

We now need two pieces of luck. First, the falling cones should be an extreme case of Re. Second, in that extreme case, the mystery function f should be easy to guess.

▶ *Are the falling cones an extreme of* Re*?*

To decide, evaluate the Re by estimating the individual factors r, v, and ν. For the speed v, everyday experience suggests – even without doing the experiment – that the cones fall at roughly $1\,\mathrm{m\,s^{-1}}$ (within a factor of 2 perhaps). Their size r is roughly $0.1\,\mathrm{m}$ (again within a factor of 2). And the kinematic viscosity of the fluid (air) in which the cones fall is $\nu \sim 10^{-5}\,\mathrm{m^2\,s^{-1}}$. [This value can be found online, in numerous handbooks such as [10, 11], or by applying approximation methods to physics and engineering calculations (the subject of another textbook).]

Combine these estimates to get the Reynolds number for a falling cone:

$$\mathrm{Re} \sim \frac{\overbrace{0.1\,\mathrm{m}}^{r} \times \overbrace{1\,\mathrm{m\,s^{-1}}}^{v}}{\underbrace{10^{-5}\,\mathrm{m^2\,s^{-1}}}_{\nu}} \sim 10^4.$$

This dimensionless number is significantly greater than 1, so the falling cones are an example of the high-Reynolds number extreme. (For the low-Reynolds number extreme, try **Problem 2.28**.)

Problem 2.22 Reynolds numbers in everyday flows
Estimate Re for:

a. a submarine cruising underwater;
b. a falling pollen grain;
c. a falling raindrop;
d. a 747 crossing the Atlantic.

Before continuing, let's catch our breath by reviewing the analysis so far. The original problem was to find the terminal velocity of a falling cone. That problem splits into two parts: (1) finding the drag as a function of speed, and (2) finding the speed at which the drag balances weight. The second problem is significantly easier than the first problem, which involves the Navier–Stokes equations of fluid mechanics.

To avoid solving those equations, use dimensions. Dimensions require that F satisfy

$$\frac{F}{\rho v^2 r^2} = f\left(\mathrm{Re}\right),$$

where $\mathrm{Re} \equiv rv/\nu$. But dimensions cannot make further progress.

Fortunately, the falling cones are an extreme case of the Reynolds number. Therefore the next step is to guess, without solving equations, how drag should behave when $\mathrm{Re} \gg 1$, then choose an $f(\mathrm{Re})$ that reproduces the behavior.

The high-Reynolds-number limit can be reached many ways. One way is to shrink the viscosity ν to 0, because ν lives in the denominator of the Reynolds number. So, in the high-Reynolds-number limit, viscosity disappears from the problem and therefore the drag should be independent of viscosity.[2]

The constraint that F be independent of ν provides enough information to guess f. To see how, look at all occurrences of ν in

$$\frac{F}{\rho v^2 r^2} = f\left(\frac{rv}{\nu}\right).$$

[2] This reasoning contains several subtle untruths, yet its conclusion is mostly correct. Clarifying the subtleties required two hundred years of mathematical and physical progress, culminating in an understanding of singular perturbations and the theory of boundary layers [12, 13].

The viscosity ν appears only on the right side via the Reynolds number. Therefore, to make F independent of ν, the right side – which is f – must be independent of Reynolds number. But f is a function only of Reynolds number, so to make it independent of Reynolds number it must be a constant! Since the left side is dimensionless, the constant must also be dimensionless:

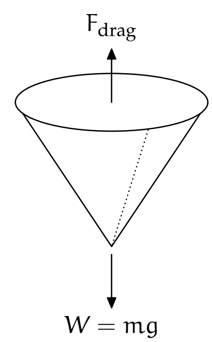
$$\frac{F}{\rho v^2 r^2} = \text{dimensionless constant.}$$

The drag force is therefore $F \sim \rho v^2 r^2$, where ρ is the density of air, v is the cone's speed, and r is its size. Since r^2 is proportional to the cone's cross-sectional area A, a common way to write the drag is

$$F \sim \rho v^2 A.$$

Although the derivation began with falling cones, the result above applies to any object moving in a fluid (as long as the Reynolds number is high). The object's shape affects only the hidden dimensionless constant. To give an idea of the typical sizes of the constant: For a sphere, it is approximately 1/4; and for a flat plate falling face down, it is roughly 1.

2.4.3 Terminal velocities



The general result gives enough information to compare the terminal velocities of the small and large cones. At its terminal velocity, a cone has no net force on it, so drag balances weight. Weight is proportional to the area of the paper used to make the cone. Therefore

$$\underbrace{\rho v^2 A}_{drag} \propto \underbrace{A}_{\text{weight}}.$$

The area divides out, so

$$v^2 \propto \rho^{-1}.$$

Since the small and large cones fall in the same fluid (air), the ρ is the same for both cones, showing that the cones fall at the same speed. In other words, *the cone's terminal velocity is independent of its size!*

To test this prediction, I made two cones as described on **page 45**, held the small and large cones above my head one in each hand, and let go.

Their fall lasted roughly two seconds, and they landed within $0.1\,\mathrm{s}$ of one another. Experiment and cheap theory agree!

Problem 2.23 Try the home experiment
Try the home experiment yourself.

Problem 2.24 Keeping more factors
The analysis in the text showed that the cone's terminal speed is independent of its size. With a little care, the analysis can predict the speed itself – not just its dependence on r. So, estimate the weight of one of the cones, predict its free-fall speed, and compare with a home experiment.

2.5 Summary and problems

The method of easy cases is based on the truism that correct solutions work in all cases – including the easy ones. Therefore, check any proposed solution in the easy cases. And guess solutions by constructing formulas that pass all easy-case tests.

Easy cases are especially powerful when combined with dimensions – the tool of **Chapter 1**. A consequence of dimensions is that any true statement about the world can be written in dimensionless form. This idea, combined with easy cases, produces the following method for guessing an unknown quantity F:

1. Find the quantities on which F depends.
2. Make dimensionless groups from among F and the quantities on which it depends.
3. Write the putative result in dimensionless form:

$$\text{dimensionless group containing } F = f(\text{other dimensionless groups}).$$

4. Use easy cases to figure out as much as possible about the dimensionless function f.

The problems are a chance to apply and extend these ideas.

Problem 2.25 Fencepost errors
A garden has 10 feet of horizontal fencing that you would like to divide into 1-foot segments. How many vertical posts do you need?

Problem 2.26 Odd sum
Here is the sum of the first n odd integers:

$$S = \underbrace{1 + 3 + 5 + \cdots + ?}_{n \text{ terms}}$$

Is the last term $2n + 1$ or $2n - 1$?

Problem 2.27 Free fall with initial velocity

The ball in **Section 1.2** was released from rest. What if it had an initial velocity v_0 (where positive v_0 means an upward throw)? Guess the impact velocity v_i.

Solve the differential equation to find the exact v_i, and compare it with your guess.

Problem 2.28 Low Reynolds number

In the limit $\text{Re} \ll 1$, guess the form of f in

$$\frac{F}{\rho v^2 r^2} = f\left(\frac{rv}{\nu}\right).$$

The result, with the correct dimensionless constant, is known as Stokes drag.

Problem 2.29 Spring equation

In the solution to the ideal-spring differential equation (**Section 1.5.2**), is it reasonable that spring constant k is in the denominator? Decide using easy cases of k.