Solution set 3

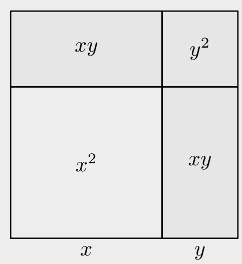
Warmups

Warmup problems are quick problems for you to check your understanding; ***don't turn them in.***

1. Draw a picture to show that

$$(x+y)^2 = x^2 + 2xy + y^2.$$

Here is the figure:



The area of the square is $(x+y)^2$, and the area divides into four regions, each corresponding to one term in the expansion

$$(x+y)^2 = x^2 + xy + xy + y^2 :$$

2. Estimate $\sqrt{26}$ by taking out the big part.

The big part is a factor of $\sqrt{25}$, so take it out (and put it back):

$$\sqrt{26} = \sqrt{25} \times \left(1 + \frac{1}{25}\right)^{1/2}.$$

Since

$$\sqrt{1+x} \approx 1 + \frac{x}{2},$$

the approximation is

$$\sqrt{26} \approx 5 \times \left(1 + \frac{1}{50}\right) = 5.1.$$

The true answer is $5.0990\ldots$, so the quick approximation is very accurate (perhaps unreasonably so).

Problems

Turn in solutions to these problems.

3. Estimate $\sqrt[3]{9}$.

> The big part is $\sqrt[3]{8}$, so
>
> $$\sqrt[3]{9} = 2 \times \sqrt[3]{1 + \frac{1}{8}}.$$
>
> Since $\sqrt[3]{1+x} \approx 1 + x/3$, the approximation is
>
> $$\sqrt[3]{9} \approx 2 + \frac{1}{12} = 2.0833\ldots.$$
>
> The true answer is 2.08008. . ..

4. Use the small-angle approximation for $\sin\theta$ to show that

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

for small θ.

> For small angles, $\sin\theta \approx \theta$. Since $\cos^2\theta + \sin^2\theta = 1$, the approximation for $\cos\theta$ is
>
> $$\cos\theta = \sqrt{1 - \sin^2\theta} \approx \sqrt{1 - \theta^2} \approx 1 - \frac{\theta^2}{2}.$$

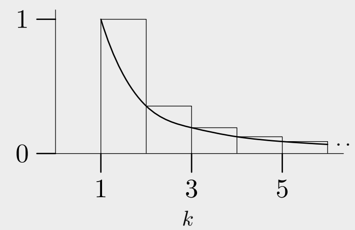
5. Riemann's zeta function

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s}$$

is important for statistical physics, for the approximate analysis of random walks, for the theory of prime numbers, and for much else. In this problem you estimate $\zeta(3/2)$, which is the sum $S = \sum_1^\infty n^{-3/2}$.

a. Sketch $f(n) = n^{-3/2}$ and, on the same diagram, draw rectangles to illustrate the sum S.

Here is an illustration:



The smooth curve is $n^{-3/2}$ and the combined area of the rectangles is the sum $\zeta(3/2)$.

b. Use the pictorial method to estimate the sum, and compare the estimate against the true value (approximately 2.612).

To approximate that area, first integrate to get the area under the smooth curve:

$$\int_1^\infty n^{-3/2}\,dn = -2n^{-1/2}\Big|_1^\infty = 2.$$

The correction is the pieces protruding beyond the curve. These pieces are almost triangles, and their total area is almost one-half the area of the first rectangle. So the correction is $1/2$ and $S \approx 2 + 0.5 = 2.5$. Compared with the more exact value $S \approx 2.61$, this estimate is low by roughly 4%. [Note: No one knows the exact values for the zeta function, except when the exponents are even integers.]

6. You want to cut a $3 \times 3 \times 3$ cube into 27 unit cubes. What is the minimum number of knife cuts that you must make? No funky knife tricks: only planar cuts!

Each cut can at best double the number of pieces. So producing 27 pieces requires at least five cuts ($2^5 = 32$ is the nearest power of two that is at least 27). Is it possible to make 27 pieces in only five cuts? Look at the resulting center cube. It has six faces, and each face requires at least one cut to produce it. So the minimum number of cuts is six. Is it possible to make 27 unit cubes in no more than six cuts? Yes: Make two parallel cuts in each coordinate direction. So six is a lower bound and an upper bound, showing that you need six cuts to make the 27 cubes.

Bonus problems

*Bonus problems are more difficult but **optional** problems for those who are curious.*

7. You want to cut a unit cube into two pieces each with volume $1/2$. What dividing surface, which might be curved, has the smallest surface area?

When bisecting the equilateral triangle, an arc of a circle centered at a vertex had the shortest path. Similarly for this problem, the octant (one-eighth) of a sphere should be the bisecting surface with the lowest area. If the cube is a unit cube, then the octant has volume $1/2$, so its radius is given by

$$\frac{1}{8} \times \frac{4}{3}\pi r^3 = \frac{1}{2}.$$

So the radius is $(3/\pi)^{1/3}$ and the surface area of the octant is

$$\text{surface area} = \frac{4\pi r^2}{8} = \frac{\pi}{2}\left(\frac{3}{\pi}\right)^{2/3} \approx 1.52.$$

Several years ago I submitted this solution to the London *Guardian* when their weekly problem contest had this three-dimensional problem.

For this solution the editors sent me a book prize. However, I realized after the prize arrived that the solution is wrong. The simplest surface – a horizontal plane through the center of the cube – has surface area 1, which is less than the surface area of the octant. (I offered to send back the book.)

The horizontal plane is the best surface that I have found, but I am not sure how to prove it. Let me know if you found a proof or a better surface!