Correlation Functions, Cluster Functions and Spacing Distributions for Random Matrices

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Abstract

The usual formulas for the correlation functions in orthogonal and symplectic matrix models express them as quaternion determinants. From this representation one can deduce formulas for spacing probabilities in terms of Fredholm determinants of matrix-valued kernels. The derivations of the various formulas are somewhat involved. In this article we present a direct approach which leads immediately to scalar kernels for the unitary ensembles and matrix kernels for the orthogonal and symplectic ensembles, and the representations of the correlation functions, cluster functions and spacing distributions in terms of them.

1. Introduction

In the most common models of random matrices the eigenvalue distribution is given by a probability density $P_N(x_1, \cdots, x_N)$. If F is a symmetric function of N variables and the eigenvalues are $\lambda_1, \cdots, \lambda_N$ then the expected value of $F(\lambda_1, \cdots, \lambda_N)$ is given by the formula

$$\int \cdots \int F(x_1, \cdots, x_N) \, P_N(x_1, \cdots, x_N) \, dx_1 \cdots dx_N.$$

Here dx denotes Lebesgue measure on a set which necessarily contains the eigenvalues ($\mathbf{R}$ if the matrices are Hermitian, $\mathbf{T}$ if the matrices are unitary, $\mathbf{C}$ if the matrices have general complex entries). The function $P_N(x_1, \cdots, x_N)$ gives the probability density that the eigenvalues lie in infinitesimal neighborhoods of $x_1, \cdots, x_N$. The n-point correlation function $R_n(x_1, \cdots, x_n)$ is defined by

$$R_n(x_1, \cdots, x_n) = \frac{N!}{(N-n)!} \int \cdots \int P_N(x_1, \cdots, x_N) \, dx_{n+1} \cdots dx_N.$$

1

It is, loosely speaking, the probability density that n of the eigenvalues, irrespective of order, lie in infinitesimal neighborhoods of $x_1, \cdots, x_n$. (It is not a probability density in the strict sense since its total integral equals $N!/(N-n)!$ rather than 1.)

Sometimes easier to compute are the n-point cluster functions $T_n(x_1, \cdots, x_n)$. These are defined in terms of the $R_m(x_1, \cdots, x_m)$ with $m \le n$. Reciprocally, the $R_n(x_1, \cdots, x_n)$ may be recovered from the $T_m(x_1, \cdots, x_m)$ with $m \le n$. These will be discussed in Section 2.

Fundamental to the study of spacings between consecutive eigenvalues is the quantity $E(0; J)$ (the E here does not represent expected value), which is the probability that the set J contain no eigenvalues. More generally $E(n_1, \cdots, n_m; J_1, \cdots, J_m)$ denotes the probability that each J_i contains precisely n_i eigenvalues. A related quantity is $P_J(x_1, \cdots, x_n)$, the probability density that the eigenvalues contained in J lie in infinitesimal intervals about $x_1, \cdots, x_n$. These are useful for the study of spacings between several eigenvalues.

All the functions R_n and T_n, as well as the just-mentioned probabilities, are obviously expressible in terms of the density function P_N. But since we generally think of N as large (indeed we are often interested in "scaling limits" as $N \to \infty$) it is desirable to find alternative expressions for these quantities which do not increase in complexity with large N, expressions in which N appears simply as a parameter.

These have been found for some particularly important matrix ensembles. They are ensembles of Hermitian matrices where P_N has the form

$$P_N(x_1, \cdots, x_N) = c_N \prod_{j<k} |x_j - x_k|^\beta \prod_j w(x_j),$$

where β equals 1, 2, or 4, $w(x)$ is a weight function and c_N is the normalization constant required to make the total integral of P_N equal to one; and ensembles of unitary matrices where P_N has the form

$$P_N(x_1, \cdots, x_N) = c_N \prod_{j<k} |e^{ix_j} - e^{ix_k}|^\beta,$$

and the integrals are taken over any interval of length 2π. The latter are Dyson's circular ensembles. (Of course one could also introduce a weight function here.) The terminology here can be confusing. The $\beta = 1, 2$ and 4 ensembles are called orthogonal, unitary and symplectic ensembles, respectively, because the underlying measures are invariant under actions of these groups. In the Hermitian case the $\beta = 1$ ensembles consist of real symmetric matrices and the $\beta = 4$ ensembles consist of self-dual Hermitian matrices. For a discussion of these matters and the analogues for the circular ensembles see [8].

Here is a rough outline of how the desired expressions were obtained when $\beta = 2$, which are the simplest cases. For Hermitian ensembles

$$P_N(x_1, \cdots, x_N) = c_N \prod_{j<k} (x_j - x_k)^2 \prod_j w(x_j).$$

On the right side we see the square of a Vandermonde determinant (for the circular ensemble it is a product of Vandermonde determinants), which is equal to the determinant of the Vandermonde matrix times its transpose. Performing row and column operations and writing

out the product, one obtains a representation

$$P_N(x_1, \cdots, x_N) = \frac{1}{N!} \det\left(K_N(x_j, x_k)\right)_{j,k=1,\cdots,N} \tag{1.1}$$

with certain kernels $K_N(x, y)$ which are expressed in terms of the polynomials orthonormal with respect to the weight function $w(x)$. These kernels are shown to satisfy a basic family of integral identities which are used to deduce that

$$R_n(x_1, \cdots, x_n) = \det\left(K_N(x_j, x_k)\right)_{j,k=1,\cdots,n}. \tag{1.2}$$

By manipulating the N-fold integral that gives $E(0; J)$ and using these expressions for the R_n one deduces that $E(0; J)$ equals the Fredholm determinant of the kernel $K_N(x, y)$ acting on J. One obtains expressions for the more general quantities $E(n_1, \cdots, n_m; J_1, \cdots, J_m)$ in terms of the same kernel. Observe that N appears only as a parameter in the kernel $K_N(x, y)$.

For β equal to 1 or 4 the situation is less simple. From the point of view of the above outline, the problem is to concoct a kernel satisfying (1.1) and the family of integral identities which led to formula (1.2). This problem was solved by Dyson [3] in the case of the circular ensembles and required the introduction of a new concept—the quaternion determinant. A quaternion-valued kernel was produced for which (1.1) and (1.2) held if all determinants were interpreted as quaternion determinants. Subsequently it was shown by Mehta [7] how to obtain analogous quaternion determinant representations for the Gaussian ensembles, the ensembles of Hermitian matrices with weight function $w(x) = e^{-x^2}$, using skew-orthogonal polynomials. This was generalized to general weights by Mehta and Mahoux [6]. (See also [8, 4, 10].) After these quaternion determinant representations for the R_n it is possible (by an argument to be found in A.7 of [8], although not explicitly stated there) to deduce representations for $E(0; J)^2$ as Fredholm determinants of 2×2 matrix kernels [11]. These matrix kernels are the matrix representations of the quaternion kernels.

If all one is interested in is the spacing distributions then this is certainly a very round-about way of obtaining Fredholm determinant representations. The purpose of this article is to present a direct approach, one which leads immediately to the scalar kernels when $\beta = 2$ and the matrix kernels when $\beta = 1$ and 4, and the representations of the correlation functions, cluster functions and spacing distributions in terms of them. It uses neither quaternion determinants nor a family of integral identities for the kernels. (That the correlation functions are equal to quaternion determinants for the $\beta = 1$ and 4 ensembles becomes a consequence of the representations.)

What we do use are the following three identities which represent certain N-fold integrals with determinant entries in terms of $N \times N$ or $2N \times 2N$ determinants with integral entries:

$$\int \cdots \int \det(\phi_j(x_k))_{j,k=1,\cdots,N} \cdot \det(\psi_j(x_k))_{j,k=1,\cdots,N} \, dx_1 \cdots dx_N$$

$$= N! \det\left(\int \phi_j(x)\,\psi_k(x)\,dx\right)_{j,k=1,\cdots,N}. \tag{1.3}$$

3

$$\int\cdots\int_{x_1\leq\cdots\leq x_N} \det(\phi_j(x_k))_{j,k=1,\cdots,N}\, dx_1\cdots dx_N$$

$$= \mathrm{Pf}\Big(\int\int \mathrm{sgn}(y-x)\,\phi_j(x)\,\phi_k(y)\,dy\,dx\Big)_{j,k=1,\cdots,N}. \tag{1.4}$$

$$\int\cdots\int \det(\phi_j(x_k)\ \ \psi_j(x_k))_{j=1,\cdots,2N,\ k=1,\cdots,N}\, dx_1\cdots dx_N$$

$$= (2N)!\,\mathrm{Pf}\Big(\int(\phi_j(x)\,\psi_k(x) - \phi_k(x)\,\psi_j(x))\,dx\Big)_{j,k=1,\cdots,2N}. \tag{1.5}$$

Here "Pf" denotes Pfaffian (its square is the determinant) and (1.4) holds for even N and must be modified for odd N. These hold for general measure spaces; in (1.4) the space must be ordered. Identities (1.4) and (1.5) are due to de Bruijn [2], who traces (1.3) as far back as 1883 [1].

In the application of (1.3) to the $\beta = 2$ case we are led to a matrix whose j, k entry equals $\delta_{j,k}$ plus the integral of a product of a function of j and a function of k. In our application of (1.4) and (1.5) to $\beta = 1$ and 4 the integrand is a sum of two such products. This, in a nutshell, is why 2×2 matrix kernels arise. What we do in each case is use one of the identities to express

$$\int\cdots\int P_N(x_1,\cdots,x_N)\prod_j(1+f(x_j))\,dx_1\cdots dx_N \tag{1.6}$$

as a determinant or Pfaffian whose entries are given by one-dimensional integrals. Manipulating the integrals leads to a scalar or matrix kernel $K_N(x,y)$ such that the above N-fold integral ($\beta = 2$) or its square ($\beta = 1$ or 4) is equal to $\det(I + K_N\, f)$. Here K_N denotes the operator with kernel $K_N(x,y)$ and f denotes multiplication by that function. Taking $f = -\chi_J$ gives immediately the representation of $E(0; J)$ or its square as $\det(I - K_N\,\chi_J)$. We also obtain representations for the more general quantities $E(n_1,\cdots,n_m; J_1,\cdots,J_m)$. Taking f to be a linear combination of delta functions leads to representations of the correlation and cluster functions in terms of the matrices $(K(x_i, x_j))$. These are 2×2 block matrices if $\beta = 1$ or 4.

All the derivations follow the same pattern. First we write the integrand in (1.6) in terms of a determinant or product of determinants. These representation are exactly as can be found in [3] or [8], for example. Then we apply whichever of (1.3)–(1.5) is appropriate, and obtain a determinant whose entries are integrals. Finally, after writing the integrand as a matrix product if necessary, we use a general identity to express the last determinant as an operator determinant.

The article is organized as follows. In section 2 we define the cluster functions and discuss their relationship with the correlation functions. In sections 3 and 4 we show how the operator determinant representations of the integrals (1.6) lead to formulas for the correlation and cluster functions as well as the spacing probabilities. In the following sections we

derive these determinant representations for the $\beta = 2$ ensembles, the $\beta = 4$ and 1 circular ensembles, and then the $\beta = 4$ and 1 Hermitian ensembles. We do things in this order since the circular ensembles are simpler than the Hermitian (because the exponentials are eigenfunctions of the differentiation operator) and $\beta = 4$ is simpler than $\beta = 1$ (because there is a single integral in (1.5) and a double integral in (1.4)). In the final section we derive the matrix kernels for the $\beta = 4$ and 1 Gaussian ensembles, which was the starting point of [11].

We emphasize that there are almost no new results in this article—perhaps the determinant formula (4.2) for the spacing probabilities and the more general form for the matrix kernels obtained in secs. 8 and 9. It is the methods used to derive them which are new and, we believe, show how they form a coherent whole.

2. Correlation and Cluster Functions

The n-point cluster functions $T_n(x_1, \cdots, x_n)$ are defined as follows. For each non-empty subset S of $\{1, \cdots, N\}$ write

$$R_S = R_n(x_{i_1}, \cdots, x_{i_n}) \quad \text{if } S = \{i_1, \cdots, i_n\}.$$

Then

$$T_n(x_1, \cdots, x_n) = \sum (-1)^{n-m} (m-1)! \, R_{S_1} \cdots R_{S_m}, \tag{2.1}$$

where the sum runs over all $m \geq 1$ and, for each m, over all partitions of $\{1, \cdots, n\}$ into nonempty subsets $S_1, \cdots, S_m$. If we know the T_n then the R_n may be recovered by use of the reciprocal formula

$$R_n(x_1, \cdots, x_n) = \sum (-1)^{n-m} T_{S_1} \cdots T_{S_m}. \tag{2.2}$$

This is most easily seen by looking at the relation between generating functions for these quantities.

Let A be a formal power series in variables $z_1, \cdots, z_N$ without constant term. For each nonempty subset S of $\{1, \cdots, N\}$ let A_S denote the coefficient of $\prod_{i \in S} z_i$ in A. Let

$$f(z) = \sum_{m=1}^{\infty} f_m \, z^m$$

be a formal power series in a single variable z without constant term, and set $B = f(A)$. For each S we have

$$B_S = \sum m! \, f_m \, A_{S_1} \cdots A_{S_m}, \tag{2.3}$$

where the sum is taken over all $m \geq 1$ and, for each m, all partitions of S into nonempty subsets $S_1, \cdots, S_m$. If $f_1 \neq 0$ then $f(z)$ has a formal inverse $g(z)$. Since $A = g(B)$ it follows from the above that for each S

$$A_S = \sum m! \, g_m \, B_{S_1} \cdots B_{S_m}. \tag{2.4}$$

We apply these fact to $f(z) = -\log(1 + z)$, $g(z) = e^{-z} - 1$, taking any formal power series A such that $A_S = R_n(x_{i_1}, \cdots, x_{i_n})$ when $S = \{i_1, \cdots, i_n\}$. Since $f_m = (-1)^m/m$ in this

5

case, we see by definition (2.1) that with the same S we have $B_S = (-1)^n T_n(x_{i_1}, \cdots, x_{i_n})$. Since $g_m = (-1)^m/m!$ (2.4) gives the reciprocal formula (2.2).

Observe that $R_n(y_1, \cdots, y_n)$ equals the coefficient of $z_1 \cdots z_n$ in the expansion about $z_1 = \cdots = z_n = 0$ of

$$\int \cdots \int P_N(x_1, \cdots, x_N) \prod_j [1 + \sum_{r=1}^{n} z_r \, \delta(x_j - y_r)] \, dx_1 \cdots dx_N. \qquad (2.5)$$

Here $(1 + \sum_{r=1}^{n} z_r \, \delta(x - y_r)) \, dx$ denotes Lebesgue measure plus masses z_r at the points y_r. Thus we may take $A(z_1, \cdots, z_n)$ to be this integral minus 1. Since $B = -\log(1 + A)$ we see, after recalling the relationship $T_S = (-1)^{|S|} B_S$, that $T_n(y_1, \cdots, y_n)$ equals $(-1)^{n+1}$ times the coefficient of $z_1 \cdots z_n$ in the expansion about $z_1 = \cdots = z_n = 0$ of

$$\log \Big(\int \cdots \int P_N(x_1, \cdots, x_N) \prod_j [1 + \sum_{r=1}^{n} z_r \, \delta(x_j - y_r)] \, dx_1 \cdots dx_N \Big). \qquad (2.6)$$

3. Formulas for the correlation and cluster functions

Suppose we have proved that for each $\beta = 2$ ensemble there is a kernel $K_N(x, y)$ such that for general f

$$\int \cdots \int P_N(x_1, \cdots, x_N) \prod_j (1 + f(x_j)) \, dx_1 \cdots dx_N = \det(I + K_N f). \qquad (3.1)$$

As before K_N denotes the operator on L^2 with kernel $K_N(x, y)$ and f as an operator denotes multiplication by that function.

Take $f(x) = \sum_{r=1}^{n} z_r \, \delta(x - y_r)$ first. Then the operator $I + K_N f$ becomes the matrix

$$\Big(\delta_{r,s} + K_N(y_r, y_s) \, z_s \Big)_{r,s=1,\cdots,n}. \qquad (3.2)$$

This may be seen by passing to a limit or observing (as we shall) that this is what happens during the derivation. It is easy to see that the coefficient of $z_1 \cdots z_n$ in the expansion of the determinant of (3.2) equals $\det \big(K_N(y_r, y_s) \big)$, and so by (2.5) we obtain formula (1.2) for the correlation functions.

For the cluster functions we use (2.6) and the general expansion

$$\log \det(I + K) = \operatorname{tr} \log(I + K) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \operatorname{tr} K^m$$

to deduce

$$T_n(y_1, \cdots, y_n) = \frac{1}{n} \sum_{\sigma} K_N(y_{\sigma 1}, y_{\sigma 2}) \cdots K_N(y_{\sigma n}, y_{\sigma 1}),$$

the sum taken over all permutations σ of $\{1, \cdots, n\}$.

6

Turning now to the $\beta = 1$ and 4 ensembles, suppose we can show that for each of them there is a matrix kernel $K_N(x, y)$ such that

$$\int \cdots \int P_N(x_1, \cdots, x_N) \prod_j (1 + f(x_j)) \, dx_1 \cdots dx_N = \sqrt{\det (I + K_N f)}. \qquad (3.3)$$

Then taking $f(x) = \sum_{r=1}^{N} z_r \, \delta(x - y_r)$ we see by (2.5) that $R_n(y_1, \cdots, y_n)$ equals the coefficient of $z_1 \cdots z_n$ in the expansion of

$$\sqrt{\det \left(\delta_{r,s} + K_N(y_r, y_s) \, z_s \right)_{r,s=1,\cdots,n}}. \qquad (3.4)$$

This coefficient is not nearly as simple as in the $\beta = 2$ case. However (2.6) shows that $T_n(y_1, \cdots, y_n)$ equals $(-1)^{n+1}$ times the coefficient of $z_1 \cdots z_n$ in the expansion of

$$\frac{1}{2} \log \det \left(\delta_{r,s} + K_N(y_r, y_s) \, z_r \right).$$

Thus we obtain

$$T_n(y_1, \cdots, y_n) = \frac{1}{2n} \sum_{\sigma} \operatorname{tr} K_N(y_{\sigma 1}, y_{\sigma 2}) \cdots K_N(y_{\sigma n}, y_{\sigma 1}),$$

which is hardly more complicated than for $\beta = 2$. To obtain formulas for the correlation functions we use (2.2) together with our expressions for the T_n. Of course the same formulas must result if we take the appropriate coefficients in (3.4).

It turns out that the 2×2 block matrices $\left(K_N(y_r, y_s) \right)$ are always self-dual. This means that the matrices on the diagonal are transposes of each other and the matrices off the diagonal are antisymmetric. Because of this the expression for the correlation function given by (2.2) shows that (1.2) holds when the determinant is interpreted as a quaternion determinant. For a discussion of this point see [8], sec. 6.2.

4. Formulas for the spacing probabilities

We obtain the formulas first assuming that $\beta = 2$ and (3.1) holds. To evaluate $E(0; J)$ we must integrate $P_N(x_1, \cdots, x_N)$ over the region where all the x_i lie in the complement of J. Thus we simply take $f = -\chi_J$ and obtain

$$E(0; J) = \det (I - K_N \chi_J).$$

More generally, to evaluate $E(n_1, \cdots, n_m; J_1, \cdots, J_m)$ we integrate $P_N(x_1, \cdots, x_N)$ over the region where n_i of the x_j lie in J_i and all the other x_j lie outside the union of the J_i. We see that $E(n_1, \cdots, n_m; J_1, \cdots, J_m)$ equals the coefficient of $(\lambda_1 + 1)^{n_1} \cdots (\lambda_m + 1)^{n_m}$ in the expansion about $\lambda_1 = \cdots = \lambda_m = -1$ of

$$\int \cdots \int P_N(x_1, \cdots, x_N) \prod_j [1 + \sum_{i=1}^{m} \lambda_i \chi_{J_i}(x_j)] \, dx_1 \cdots dx_N.$$

Therefore we take $f(x) = \sum_{i=1}^{m} \lambda_i \, \chi_{J_i}(x)$ in (3.1) and deduce that

$$E(n_1, \cdots, n_m; J_1, \cdots, J_m)$$

$$= \frac{1}{n_1! \cdots n_m!} \frac{\partial^{\sum n_i}}{\partial \lambda_1^{n_1} \cdots \partial \lambda_m^{n_m}} \det \left(I + K_N \sum \lambda_i \, \chi_{J_i} \right) \Big|_{\lambda_1 = \cdots = \lambda_m = -1}. \tag{4.1}$$

To see the connection with spacings between eigenvalues, consider the quantity

$$[E(0; (x + \Delta x, y)) - E(0; (x, y))] - [E(0; (x + \Delta x, y + \Delta y)) - E(0; (x, y + \Delta y))],$$

where Δx and Δy are positive. The difference on the left equals the probability that there is no eigenvalue in $(x + \Delta x, y)$ but there is an eigenvalue in the larger interval (x, y), which is the same as saying that there is no eigenvalue in $(x + \Delta x, y)$ but there is an eigenvalue in $(x, x + \Delta x)$. Similarly the difference on the right equals the probability that there is no eigenvalue in $(x + \Delta x, y + \Delta y)$ but there is an eigenvalue in $(x, x + \Delta x)$. Therefore the difference equals the probability that there is no eigenvalue in $(x + \Delta x, y)$ but there is an eigenvalue in both $(x, x + \Delta x)$ and $(y, y + \Delta x)$. Hence if we divide this difference by $\Delta x \, \Delta y$ and take the limit as $\Delta x, \, \Delta y \to 0$ we obtain the joint probability density for consecutive eigenvalues to lie in infinitesimal intervals about x and y. On the other hand this same limit equals

$$-\frac{d^2}{dy \, dx} E(0; (x, y)).$$

Therefore this is the formula for the joint probability density, and explains why the spacing distributions are intimately connected with Fredholm determinants.

To obtain the conditional probability density that, given an eigenvalue at x, the next eigenvalue lies in an infinitesimal neighborhood of y, we must divide the above by the probability density that there is an eigenvalue in an infinitesimal neighborhood of x. This is the 1-point correlation function $R_1(x, x)$ which, we know, equals $K_N(x, x)$. Therefore this conditional probability density equals

$$-\frac{1}{K_N(x, x)} \frac{d^2}{dy \, dx} E(0; (x, y)).$$

We now compute $P_J(x_1, \cdots, x_n)$, the joint probability density that the eigenvalues contained in an interval J lie in infinitesimal intervals about $x_1, \cdots, x_n$. We observe first that the probability that a small interval contains more than one eigenvalue goes to zero as the square of the length of the interval. So we shall apply (4.1) to small intervals about the x_i (taking these $n_i = 1$) and the intervals constituting the rest of J (taking these $n_i = 0$), divide by the product of the lengths of the small intervals and let their lengths tend to 0. We denote the small intervals by $J_1, \cdots, J_n$ and the rest of J by J'. Denote by $E_{J_1, \cdots, J_n}$ the probability that each J_i contains one eigenvalue and J' none. By (4.1) we have

$$E_{J_1, \cdots, J_n} = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \det \left(I - K_N \, \chi_{J'} + K_N \sum_{i=1}^{n} \lambda_i \, \chi_{J_i} \right) \Big|_{\lambda_1 = \cdots = \lambda_n = -1}$$

8

$$= \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det \left(I - K_N \, \chi_J + K_N \sum_{i=1}^n z_i \, \chi_{J_i} \right) \Big|_{z_1 = \cdots = z_n = 0}.$$

We factor the operator in this expression as

$$(I - K_N \, \chi_J)\,(I + R_J \sum_{i=1}^n z_i \, \chi_{J_i}),$$

where R_J is the resolvent kernel of $K_N \, \chi_J$, the kernel of $(I - K_N \, \chi_J)^{-1} \, K_N \, \chi_J$. Determinants multiply, and the determinant of the first factor equals $E(0; J)$, as we know. Thus

$$E_{J_1, \cdots, J_n} = E(0; J) \, \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det \left(I + R_J \sum_{i=1}^n z_i \, \chi_{J_i} \right) \Big|_{z_1 = \cdots = z_n = 0}.$$

Now we can proceed as for the correlation and cluster functions. If

$$A(z_1, \cdots, z_n) = \det \left(I + R_J \sum_{i=1}^n z_i \, \chi_{J_i} \right) - 1,$$

$$B(z_1, \cdots, z_n) = \log \det \left(I + R_J \sum_{i=1}^n z_i \, \chi_{J_i} \right),$$

then $E_{J_1, \cdots, J_n}/E(0; J)$ equals the coefficient of $z_1 \cdots z_n$ in the expansion of A about $z_1 = \cdots = z_n = 0$ and $A = e^B - 1$. The coefficients of A are determined from the coefficients of B by formula (2.4), which gives in this case

$$A_S = \sum B_{S_1} \cdots B_{S_m},$$

where the sum is taken over all partitions $\{S_1, \cdots, S_m\}$ of S into nonempty subsets. To evaluate

$$B_S = \prod_{i \in S} \frac{\partial}{\partial z_i} \log \det \left(I + R_J \sum_{i \in S} z_i \, \chi_{J_i} \right) \Big|_{z_1 = \cdots = z_n = 0}$$

we use the general facts for an operator function $K(z)$ that

$$\frac{d}{dz} \log \det K(z) = \operatorname{tr} K(z)^{-1} \, K'(z), \quad \frac{d}{dz} K(z)^{-1} = -K(z)^{-1} \, K'(z) \, K(z)^{-1}.$$

We apply this consecutively to the derivatives with respect to the z_i and find that if $S = \{i_1, \cdots, i_k\}$ then

$$B_S = \frac{(-1)^{k-1}}{k} \sum_\sigma \operatorname{tr} R_J \, \chi_{J_{i_{\sigma 1}}} \cdots R_J \, \chi_{J_{i_{\sigma k}}},$$

where σ runs over all permutations of $\{1, \cdots, k\}$.

Eventually we have to divide the various products $B_{S_1} \cdots B_{S_m}$ by the product of the lengths of all the J_i and take the limit as these lengths tend to 0. We may consider separately each B_S and its corresponding intervals since the S_i are disjoint. So we compute

$$\operatorname{tr} R_J \, \chi_{J_{i_1}} \cdots R_J \, \chi_{J_{i_k}} = \int_{J_{i_k}} \cdots \int_{J_{i_1}} R_J(x_{i_k}, x_{i_1}) \, R_J(x_{i_1}, x_{i_2}) \cdots R_J(x_{i_{k-1}}, x_{i_k}) \, dx_{i_1} \cdots dx_{i_k}.$$

If $J_{i_1}, \cdots, J_{i_k}$ are small intervals about $x_{i_1}, \cdots, x_{i_k}$ then dividing by the product of their lengths and letting the lenths tend to 0 gives in the limit

$$R_J(x_{i_k}, x_{i_1}) \, R_J(x_{i_1}, x_{i_2}) \cdots R_J(x_{i_{k-1}}, x_{i_k}).$$

Now we would have obtained exactly the same results if, instead of starting with the determinant of the operator $I + R_J \sum z_i \, \chi_{J_i}$, differentiating with respect to the z_i, setting all $z_i = 0$, dividing and taking a limit, we had simply started with the determinant of the matrix

$$\left(\delta_{i,j} + R_J(x_i, x_j) \, z_j \right)_{i,j=1,\cdots,n}$$

and differentiated with respect to the z_i and set all $z_i = 0$. It follows that the the ratio $P_J(x_1, \cdots, x_n)/E(0; J)$ equals

$$\frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det \left(\delta_{i,j} + R_J(x_i, x_j) \, z_j \right) \Big|_{z_1 = \cdots = z_n = 0}.$$

Therefore we have obtained the formula

$$P_J(x_1, \cdots, x_n) = E(0; J) \det \left(R_J(x_i, x_j) \right)_{i,j=1\cdots n}. \tag{4.2}$$

Note the similarity to formula (1.2) for the correlation functions: to obtain the formula for the ratio $P_J(x_1, \cdots, x_n)/E(0; J)$ we replace the kernel $K_N(x, y)$ by the kernel $R_J(x, y)$.

We mention that rather more complicated formulas for these probability densities were obtained by Mehta and des Cloizeaux [9]. (See also [8].) These involve the eigenvalues and eigenfunctions of the operator $K_N \chi_J$ but they must must be equivalent to (4.2).

For the spacing probabilities when $\beta = 1$ or 4 we must make modifications as at the end of the previous section. We have in this case

$$E(0; J)^2 = \det \left(I - K_N \, \chi_J \right)$$

and the ratio $P_J(x_1, \cdots, x_n)/E(0; J)$ is again given by the same formulas as for the correlation functions but with $K_N(x, y)$ replaced by $R_J(x, y)$. One can show that the block matrices $(R_J(x_i, x_j))$ that arise here are also self-dual, and this implies that the formula for $P_J(x_1, \cdots, x_n)$ is as above if the determinant is interpreted as a quaternion determinant.

5. The $\beta = 2$ matrix ensembles

Consider the Hermitian case first. We use (1.3) to express

$$\int \cdots \int \prod_{j<k} (x_j - x_k)^2 \prod_j w(x_j) \prod_j (1 + f(x_j)) \, dx_1 \cdots dx_N \tag{5.1}$$

as a determinant. Since

$$\prod_{j<k} (x_j - x_k) = \det \, (x_k^j)_{j=0,\cdots,N-1, \; k=1,\cdots,N}$$

10

applying (1.3) with $\phi_j(x) = \psi_j(x) = x^j$ and with the measure $w(x)\,(1 + f(x))\,dx$ shows that this equals $N!$ times

$$\det \left(\int x^{j+k}\, w(x)\,(1 + f(x))\,dx \right)_{j,k=0,\cdots,N-1}.$$

If we denote by $\{\varphi_j(x)\}$ the sequence obtined by orthonormalizing the sequence $\{x^j\, w(x)^{\frac{1}{2}}\}$ then we see that the above is equal, except for a different constant factor depending only on N, to

$$\det \left(\int \varphi_j(x)\, \varphi_k(x)\,(1 + f(x))\,dx \right)_{j,k=0,\cdots,N-1}$$

$$= \det \left(\delta_{j,k} + \int \varphi_j(x)\, \varphi_k(x)\, f(x)\,dx \right)_{j,k=0,\cdots,N-1}. \tag{5.2}$$

Now we use something which is needlessly fancy but which is very useful, namely the general relation $\det(I + AB) = \det(I + BA)$ for arbitrary Hilbert-Schmidt operators A and B. They may act between different spaces as long as the products make sense. In our case we take A to be the operator from L^2 to $\mathbf{C}^N$ with kernel $A(j,x) = \varphi_j(x)\, f(x)$ and B the operator from $\mathbf{C}^N$ to L^2 with kernel $B(x,j) = \varphi_j(x)$. Then

$$AB(j,k) = \int \varphi_j(x)\, \varphi_k(x)\, f(x)\,dx,$$

so (5.2) equals $\det(I + AB)$. Now BA is the operator on L^2 with kernel $K_N(x,y)\, f(y)$ where

$$K_N(x,y) = \sum_{k=0}^{N-1} \varphi_k(x)\, \varphi_k(y).$$

Therefore (5.2) also equals $\det(I + K_N f)$.

Hence we have shown for these ensembles that (3.1) holds up to a constant factor depending only on N. In fact the constant factor now must be 1, as is seen by setting $f = 0$, and so (3.1) is true as it stands.

It is easily seen directly that the nonzero eigenvalues of $K_N f$ are the same as those of the matrix $(\int \varphi_j(x)\, \varphi_k(x)\, f(x)\,dx)$, for any eigenfunction must be a linear combination of the φ_j and solving for the coefficients leads directly to the matrix. So the fact $\det(I + AB) = \det(I + BA)$ is very simple here.

When the measure $f(x)\,dx$ is discrete, in other words when $f(x) = \sum_{r=1}^{n} z_r\, \delta(x - y_r)$, the integral in the right side of (5.2) is replaced by $\sum_{r=1}^{n} z_r\, \varphi_j(y_r)\, \varphi_k(y_r)$. If we set

$$A(j,r) = z_r\, \varphi_j(y_r), \quad B(r,j) = \varphi_j(y_r), \qquad (j = 0, \cdots, N-1, \quad r = 1, \cdots, n)$$

then our matrix is AB, and

$$BA(r,s) = K_N(y_r, y_s)\, z_s, \qquad (r, s = 1, \cdots, n)$$

with K_N as before. Thus in (3.1) the operator $I + K_N f$ is replaced by the matrix (3.2), as claimed. (The same argument will hold for the other ensembles, and there will be no need to repeat it.)

11

For the circular $\beta = 2$ ensemble (5.1) is replaced by

$$\int \cdots \int \prod_{j<k}(e^{-ix_j} - e^{-ix_k}) \prod_{j<k}(e^{ix_j} - e^{ix_k}) \prod_j (1 + f(x_j))\, dx_1 \cdots dx_N.$$

In the application of (1.3) we take $\phi_j(x) = e^{-ix_j}/\sqrt{2\pi}$, $\psi_j(x) = e^{ix_j}/\sqrt{2\pi}$ so that the analogue of the matrix in (5.2) becomes

$$\frac{1}{2\pi}\int e^{i(k-j)x} f(x)\, dx.$$

(The $1/2\pi$ factor is needed to obtain $\delta_{j,k}$ when $f = 0$.) Now we set

$$A(j,x) = \frac{1}{2\pi}e^{-i(j - \frac{N-1}{2})x} f(x), \quad B(x,j) = e^{i(j - \frac{N-1}{2})x},$$

(the extra exponential factors yield a simpler formula), so that the above matrix equals AB while BA equals the operator with kernel $K_N(x,y)\, f(y)$ where now

$$K_N(x,y) = \frac{1}{2\pi}\sum_{k=0}^{N-1} e^{i(k - \frac{N-1}{2})(x-y)} = \frac{1}{2\pi}\frac{\sin\frac{N}{2}(x-y)}{\sin\frac{1}{2}(x-y)}.$$

Thus (3.1) holds for the circular ensemble with this replacement for K_N.

6. The $\beta = 4$ circular ensemble

We begin with the evaluation of

$$\int \cdots \int \prod_{j<k} |e^{ix_j} - e^{ix_k}|^4 \prod_j (1 + f(x_j))\, dx_1 \cdots dx_N \tag{6.1}$$

as a Pfaffian. It is a pretty fact that

$$\prod_{j<k}(x_j - x_k)^4 = \det\left(x_k^j \quad j x_k^{j-1}\right)_{j=0,\cdots,2N-1,\ k=1,\cdots,N}. \tag{6.2}$$

This is seen by writing the product representation of the Vandermonde determinant

$$\det\left(x_k^j \quad y_k^j\right)_{j=0,\cdots,2N-1,\ k=1,\cdots,N}$$

then differentiating with respect to each y_k and setting $y_k = x_k$.

If we replace each x_k by e^{ix_k} and use the fact that

$$|e^{ix_j} - e^{ix_k}| = i\, e^{-i\frac{x_j + x_k}{2}}\left(e^{ix_j} - e^{ix_k}\right) \tag{6.3}$$

if $x_j \leq x_k$ (both lying in the same interval of length 2π) we see that

$$\prod_{j<k}|e^{ix_j} - e^{ix_k}|^4 = e^{-2i(N-1)\sum x_j} \prod_{j<k}(e^{ix_j} - e^{ix_k})^4.$$

12

By (6.2) this is equal to

$$e^{-2i(N-1)\sum x_j} \det \left(e^{ijx_k} \quad je^{i(j-1)x_k} \right)_{j=0,\cdots,2N-1, \ k=1,\cdots,N} = \det \left(e^{i(j-N+\frac{1}{2})x_k} \quad je^{i(j-N+\frac{1}{2})x_k} \right)$$

$$= \det \left(e^{i(j-N+\frac{1}{2})x_k} \quad (j-N+\tfrac{1}{2})e^{i(j-N+\frac{1}{2})x_k} \right) = \det \left(e^{ipx_k} \quad pe^{ipx_k} \right),$$

where in the last determinant $k = 1, \cdots, N$ as before but p runs through the half integers $-N + \frac{1}{2}, -N + \frac{3}{2}, \cdots, N - \frac{1}{2}$.

By the symmetry of the integrand in (6.1) the integral equals $N!$ times the integral over this region $x_1 \le \cdots \le x_N$. So we can use formula (1.5) to deduce that the square of (6.1) equals a constant depending only on N times the $2N \times 2N$ determinant

$$\det \left(\int (q-p) \, e^{i(p+q)x} \, (1 + f(x)) \, dx \right).$$

Both indices p and q run over the half-integers $-N + \frac{1}{2}, \cdots, N - \frac{1}{2}$. If we reverse the order of the rows and divide each column by its index q we see that this determinant is equal to another constant depending only on N times

$$\det \left(\frac{1}{4\pi} \int (1 + \frac{p}{q}) \, e^{i(-p+q)x} \, (1 + f(x)) \, dx \right) = \det \left(\delta_{p,q} + \frac{1}{4\pi} \int (1 + \frac{p}{q}) \, e^{i(-p+q)x} \, f(x) \, dx \right).$$

Here we see the sum of products $e^{-ipx} e^{iqx} + p \, e^{-ipx} \frac{1}{q} e^{iqx}$ referred to in the introduction, and we write it as a matrix product

$$\left(\begin{array}{cc} e^{-ipx} & ipe^{-ipx} \end{array} \right) \left(\begin{array}{c} e^{iqx} \\ \frac{1}{iq} e^{iqx} \end{array} \right).$$

(The reason for the insertions of i will become apparent.) Thus if we set

$$A(p,x) = \frac{1}{4\pi} f(x) \left(\begin{array}{cc} e^{-ipx} & ip \, e^{-ipx} \end{array} \right), \quad B(x,q) = \left(\begin{array}{c} e^{iqx} \\ \frac{1}{iq} e^{iqx} \end{array} \right)$$

then the above matrix is $I + AB$. In this case BA is the integral operator with matrix kernel $K_N(x,y) \, f(y)$ where

$$K_N(x,y) = \left(\begin{array}{cc} \frac{1}{4\pi} \sum e^{ip(x-y)} & \frac{1}{4\pi} \sum ip \, e^{ip(x-y)} \\ \frac{1}{4\pi} \sum \frac{1}{ip} e^{ip(x-y)} & \frac{1}{4\pi} \sum e^{ip(x-y)} \end{array} \right).$$

If we write

$$S_N(x) = \frac{1}{2\pi} \sum_p e^{ipx} = \frac{1}{2\pi} \frac{\sin Nx}{\sin \frac{1}{2}x}, \quad DS_N(x) = \frac{d}{dx} S_N(x), \quad IS_N(x) = \int_0^x S_N(y) \, dy,$$

then

$$K_N(x,y) = \frac{1}{2} \left(\begin{array}{cc} S_N(x-y) & DS_N(x-y) \\ IS_N(x-y) & S_N(x-y) \end{array} \right). \tag{6.4}$$

13

Note the relationship between $S_N(x-y)$ and the kernel $K_N(x,y)$ which arises in the $\beta = 2$ ensemble: the former equals the latter with N replaced by $2N$.

Recall that we have been computing, not the integral (6.1), but its square. Thus we have shown for this ensemble and with this K_N that (3.3) holds up to a constant factor depending only on N. As before, taking $f = 0$ shows that the constant factor equals 1.

7. The $\beta = 1$ circular ensemble

We assume here that N is even and begin with the evaluation of

$$\int \cdots \int \prod_{j<k} |e^{ix_j} - e^{ix_k}| \prod_j (1 + f(x_j)) \, dx_1 \cdots dx_N. \tag{7.1}$$

Using (6.3) we see that in the region $x_1 \leq \cdots \leq x_N$,

$$\prod_{j<k} |e^{ix_j} - e^{ix_k}| = i^{-N(N-1)} e^{-i\frac{N-1}{2}\sum x_j} \prod_{j<k} (e^{ix_k} - e^{ix_j})$$

$$= i^{-N(N-1)} e^{-i\frac{N-1}{2}\sum x_j} \det \left(e^{ijx_k}\right)_{j=0,\cdots,N-1,\ k=1,\cdots,N} = i^{-N(N-1)} \det \left(e^{ipx_k}\right),$$

where in the last determinant $k = 1, \cdots, N$ and p runs through the half integers from $-\frac{N}{2}+\frac{1}{2}$ to $\frac{N}{2}-\frac{1}{2}$, as before but with N replaced by $N/2$. The i-factor in the last term equals ± 1. We now apply (1.4) with the measure $(1 + f(x)) \, dx$, using the fact that $\prod |e^{ix_j} - e^{ix_k}|$ is a symmetric function, and conclude that the square of (7.1) is a constant depending only on N times

$$\det \left(\int \int \operatorname{sgn}(y - x) \, e^{ipx} \, e^{iqy} \, (1 + f(x)) \, (1 + f(y)) \, dy \, dx \right).$$

First, we replace p by $-p$ as before. Then we set $\varepsilon(x) = \frac{1}{2}\operatorname{sgn}(x)$, and replace $\operatorname{sgn}(y-x)$ by $\varepsilon(x-y)$ in the determinant. These things just change the multiplying constant. We shall also use the notation $e_p(x)$ for the normalized exponential $e^{ipx}/\sqrt{2\pi}$. With these modifications the above becomes

$$\det \left(\int \int \varepsilon(x - y) \, e_{-p}(x) \, e_q(y) \, (1 + f(x)) \, (1 + f(y)) \, dy \, dx \right).$$

Denote by ε the operator with kernel $\varepsilon(x - y)$. Observe that ε is antisymmetric and $\varepsilon e_p = e_p/ip$. (The latter uses the fact that p is an integer plus $\frac{1}{2}$.) Writing

$$(1 + f(x)) \, (1 + f(y)) = 1 + f(x) + f(y) + f(x) \, f(y)$$

and using the fact that ε is antisymmetric, we see that the double integral equals

$$\frac{1}{iq}\delta_{p,q} + \int \left[\frac{1}{iq} f(x) \, e_{-p}(x) \, e_q(x) \, dx + \frac{1}{ip} f(x) \, e_{-p}(x) \, e_q(x) - f(x) \, e_q(x) \, \varepsilon(f e_{-p})(x)\right] dx,$$

so the determinant equals another constant depending on N times that of the matrix with p, q entry

$$\delta_{p,q} + \int \left[f(x) \, e_{-p}(x) \, e_q(x) + \frac{q}{p} f(x) \, e_{-p}(x) \, e_q(x) - iq \, f(x) \, e_q(x) \, \varepsilon(f e_{-p})(x)\right] dx.$$

At this point setting $f = 0$ shows that the multiplying constant equals 1.

We write the integrand in the p, q entry of the matrix as a matrix product

$$\left(\begin{array}{cc} \frac{1}{ip} f(x) e_{-p}(x) - f(x) \varepsilon(f e_{-p})(x) & f(x) e_{-p}(x) \end{array} \right) \left(\begin{array}{c} iq\, e_q(x) \\ \\ e_q(x) \end{array} \right)$$

and apply $I + AB \to I + BA$ as before to replace this by a matrix kernel. For simplicity of notation we shall write out the operator rather than the kernel and use the notation $a \otimes b$ for the operator with kernel $a(x)\, b(y)$. The operator $I + BA$ is equal to

$$\left(\begin{array}{cc} I + \sum e_p \otimes f e_{-p} - \sum ip\, e_p \otimes f\, \varepsilon(f e_{-p}) & \sum ip\, e_p \otimes f e_{-p} \\ \\ \sum \frac{1}{ip} e_p \otimes f e_{-p} - \sum e_p \otimes f\, \varepsilon(f e_{-p}) & I + \sum e_p \otimes f e_{-p}. \end{array} \right)$$

Now it is an easy fact that for an operator A we have $(a \otimes b)\, A = a \otimes (A^t\, b)$. Therefore the second sums in the entries of the first column of the matrix, with their minus signs, equal the corresponding entries of the second column right-multiplied by the operator $\varepsilon\, f$. (Here f denotes multiplication by the function, which is symmetric, and we used the fact that ε is antisymmetric.) It follows that the matrix may be written as the product

$$\left(\begin{array}{cc} I + \sum e_p \otimes f e_{-p} & \sum ip\, e_p \otimes f e_{-p} \\ \\ \sum \frac{1}{ip} e_p \otimes f e_{-p} - \varepsilon f & I + \sum e_p \otimes f e_{-p} \end{array} \right) \left(\begin{array}{cc} I & 0 \\ \\ \varepsilon f & I \end{array} \right).$$

The operator on the right has determinant 1 and so the determinant of the product equals the determinant of the operator on the left, i.e., the determinant of

$$I + \left(\begin{array}{cc} \sum e_p \otimes e_{-p} & \sum ip\, e_p \otimes e_{-p} \\ \\ \sum \frac{1}{ip} e_p \otimes e_{-p} - \varepsilon & \sum e_p \otimes e_{-p} \end{array} \right) f.$$

If we recall the definition of e_p and the notation of the last section we see that (3.3) holds for this ensemble when K_N has kernel

$$K_N(x, y) = \left(\begin{array}{cc} S_{\frac{N}{2}}(x - y) & DS_{\frac{N}{2}}(x - y) \\ \\ IS_{\frac{N}{2}}(x - y) - \varepsilon(x - y) & S_{\frac{N}{2}}(x - y). \end{array} \right)$$

8. The $\beta = 4$ Hermitian ensembles

We begin now with the N-fold integral

$$\int \cdots \int \prod_{j<k} (x_j - x_k)^4 \prod_j w(x_j) \prod_j (1 + f(x_j))\, dx_1 \cdots dx_N.$$

15

Using (6.2) as it stands and (1.5) we see that the square of this N-fold integral equals a constant depending only on N times

$$\det \left(\int (k-j)\, x^{j+k-1}\, (1+f(x))\, w(x)\, dx \right)_{j,k=0,\cdots,2N-1}.$$

We replace the sequence $\{x^j\}$ by any sequence $\{p_j(x)\}$ of polynomials of exact degree j. Except for another constant factor depending only on N, the above equals

$$\det \left(\int (p_j(x)\, p_k'(x) - p_j'(x)\, p_k(x))\, (1+f(x))\, w(x)\, dx \right).$$

Just as when $\beta = 2$ it was convenient to introduce functions φ_j which were equal to the orthonormal polynomials times $w^{\frac{1}{2}}$ so now we introduce $\psi_j = p_j\, w^{\frac{1}{2}}$. The matrix is equal to

$$\left(\int (\psi_j(x)\, \psi_k'(x) - \psi_j'(x)\, \psi_k(x))\, (1+f(x))\, dx \right),$$

the extra terms arising from the differentiations having cancelled. We write this as

$$M + \left(\int (\psi_j(x)\, \psi_k'(x) - \psi_j'(x)\, \psi_k(x))\, f(x)\, dx \right)$$

where M is the matrix of integrals $\int (\psi_j(x)\, \psi_k'(x) - \psi_j'(x)\, \psi_k(x))\, dx$.

Next we factor out M, say on the left. Its determinant is just another constant depending only on N. If $M^{-1} = (\mu_{jk})$ and we set $\eta_j(x) = \sum \mu_{jk}\, \psi_k(x)$ then the resulting matrix is

$$I + \left(\int (\eta_j(x)\, \psi_k'(x) - \eta_j'(x)\, \psi_k(x))\, f(x)\, dx \right),$$

Next, we write the sum of products $\eta_j(x)\, \psi_k'(x) - \eta_j'(x)\, \psi_k(x)$ as a matrix product

$$\begin{pmatrix} \eta_j(x) & -\eta_j'(x) \end{pmatrix} \begin{pmatrix} \psi_k'(x) \\ \psi_k(x) \end{pmatrix}.$$

Thus if we set

$$A(j,x) = f(x) \begin{pmatrix} \eta_j(x) & -\eta_j'(x) \end{pmatrix}, \quad B(x,j) = \begin{pmatrix} \psi_j'(x) \\ \psi_j(x) \end{pmatrix},$$

then the above matrix is $I + AB$. In this case BA is the integral operator with matrix kernel $K_N(x,y)\, f(y)$ where

$$K_N(x,y) = \begin{pmatrix} \sum \psi_j'(x)\, \eta_j(y) & -\sum \psi_j'(x)\, \eta_j'(y) \\ \sum \psi_j(x)\, \eta_j(y) & -\sum \psi_j(x)\, \eta_j'(y) \end{pmatrix}.$$

The sums here are over $j = 0, \cdots, 2N-1$.

16

We have shown that with this K_N (3.3) holds up to a constant factor depending on N and as usual taking $f = 0$ shows that the constant factor equals 1.

Recall the definition of the η_j. If we write

$$S_N(x, y) = 2 \sum_{j,k=0}^{2N-1} \psi'_j(x) \, \mu_{jk} \, \psi_k(y)$$

and

$$IS_N(x, y) = 2 \sum_{j,k=0}^{2N-1} \psi_j(x) \, \mu_{jk} \, \psi_k(y), \quad S_N D(x, y) = -2 \sum_{j,k=0}^{2N-1} \psi'_j(x) \, \mu_{jk} \, \psi'_k(y)$$

then

$$K_N(x, y) = \frac{1}{2} \begin{pmatrix} S_N(x, y) & S_N D(x, y) \\ IS_N(x, y) & S_N(y, x) \end{pmatrix}. \tag{8.1}$$

The explanation for the notation is that if S_N is the operator with kernel $S_N(x, y)$ then $S_N D(x, y)$ is the kernel of $S_N D$ (D = differentiation) and $IS_N(x, y)$ is the kernel of IS_N (I = integration, more or less). The factors 2 and $\frac{1}{2}$ are inserted to maintain the analogy with (6.4).

One might wonder about the choice of the p_j and the resulting μ_{jk}. Since M is anti-symmetric the formulas would be simplest if it were the direct sum of N copies of the 2×2 matrix $Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, whose inverse equals its negative. The polynomials that achieve this are called skew-orthogonal. But there is no actual necessity for using these since, as is easily seen, any family of polynomials leads to the same matrix kernel.

9. The $\beta = 1$ Hermitian ensembles

Again we assume that N is even and begin now with

$$\int \cdots \int \prod_{j<k} |x_j - x_k| \prod_j w(x_j) \prod_j (1 + f(x_j)) \, dx_1 \cdots dx_N.$$

This is equal to $N!$ times the integral over the region $x_1 \leq \cdots \leq x_N$ and each $|x_j - x_k| = x_k - x_j$ there. Using (1.4) we see therefore that the square of the integral equals a constant depending only on N times

$$\det \left(\int \int \varepsilon(x - y) \, x^j \, y^k \, (1 + f(x)) \, (1 + f(y)) \, w(x) \, w(y) \, dy \, dx \right)_{j,k=0,\cdots,N-1}.$$

As in the last section we replace $\{x^j\}$ by an arbitrary sequence of polynomials $\{p_j(x)\}$ of exact degree j.. The above becomes a constant depending only on N times

$$\det \left(\int \int \varepsilon(x - y) \, p_j(x) \, p_k(y) \, (1 + f(x) + f(y) + f(x) \, f(y)) \, w(x) \, w(y) \, dy \, dx \right).$$

Now set $\psi_j(x) = p_j(x)\, w(x)$ and denote by M the matrix with j, k entry

$$\int \int \varepsilon(x - y)\, p_j(x)\, p_k(y)\, w(x)\, w(y)\, dy\, dx.$$

Then the last determinant equals

$$\det \left(M + \int \int \varepsilon(x - y)\, \psi_j(x)\, \psi_k(y)\, (1 + f(x) + f(y) + f(x)\, f(y))\, dy\, dx \right)$$

$$= \det \left(M + \int [f\, \psi_j\, \varepsilon\psi_k - f\, \psi_k\, \varepsilon\psi_j - f\, \psi_k\, \varepsilon(f\psi_j)]\, dx \right).$$

All expressions in the integrand are functions of x. As earlier, the operator with kernel $\varepsilon(x - y)$ is denoted by ε.

As in the last section, we factor out M on the left, write $M^{-1} = (\mu_{jk})$ and set $\eta_j = \sum \mu_{jk}\, \psi_k$. What results is the determinant of

$$I + \left(\int [f\, \eta_j\, \varepsilon\psi_k - f\, \psi_k\, \varepsilon\eta_j - f\, \psi_k\, \varepsilon(f\eta_j)]\, dx \right),$$

and, as usual, at this point the multiplying constant equals 1.

The integral is equal to the j, k entry of the product AB, where

$$A(j, x) = \left(-f\, \varepsilon\eta_j - f\, \varepsilon(f\eta_j) \quad f\, \eta_j \right), \quad B(x, j) = \begin{pmatrix} \psi_j \\ \varepsilon\psi_j \end{pmatrix}.$$

The operator $I + BA$ equals

$$\begin{pmatrix} I - \sum \psi_j \otimes f\, \varepsilon\eta_j - \sum \psi_j \otimes f\, \varepsilon(f\eta_j) & \sum \psi_j \otimes f\, \eta_j \\ -\sum \varepsilon\psi_j \otimes f\, \varepsilon\eta_j - \sum \varepsilon\psi_j \otimes f\, \varepsilon(f\eta_j) & I + \sum \varepsilon\psi_j \otimes f\, \eta_j \end{pmatrix}.$$

As in the circular ensemble, we can write this as the matrix product

$$\begin{pmatrix} I - \sum \psi_j \otimes f\, \varepsilon\eta_j & \sum \psi_j \otimes f\, \eta_j \\ -\sum \varepsilon\psi_j \otimes f\, \varepsilon\eta_j - \varepsilon f & I + \sum \varepsilon\psi_j \otimes f\, \eta_j \end{pmatrix} \begin{pmatrix} I & 0 \\ \varepsilon f & I \end{pmatrix}.$$

The determinant of the product equals the determinant of the first factor, i.e., the determinant of

$$I + \begin{pmatrix} -\sum \psi_j \otimes \varepsilon\eta_j & \sum \psi_j \otimes \eta_j \\ -\sum \varepsilon\psi_j \otimes \varepsilon\eta_j - \varepsilon & \sum \varepsilon\psi_j \otimes \eta_j \end{pmatrix} f.$$

Recalling the definition of the q_j we now define

$$S_N(x, y) = -\sum_{j,k=0}^{N-1} \psi_j(x)\, \mu_{jk}\, \varepsilon\psi_k(y),$$

18

$$IS_N(x,y) = -\sum_{j,k=0}^{N-1} \varepsilon\psi_j(x)\,\mu_{jk}\,\varepsilon\psi_k(y), \quad S_N D(x,y) = \sum_{j,k=0}^{N-1} \psi_j(x)\,\mu_{jk}\,\psi_k(y),$$

and we see that the (3.3) holds for this ensemble when K_N has kernel

$$K_N(x,y) = \begin{pmatrix} S_N(x,y) & S_N D(x,y) \\ IS_N(x,y) - \varepsilon(x-y) & S_N(y,x) \end{pmatrix}. \tag{9.1}$$

Once again the matrix kernel is independent of the choice of the p_j but the expression is simplest when the polynomials are skew-orthogonal in this context, i.e., when M is the direct sum of $N/2$ copies of Z.

10. The Gaussian ensembles

The Gaussian symplectic ensemble ($\beta = 4$). The weight function $w(x)$ now equals e^{-x^2}. As in sec. 5 we use the notation φ_j for the polynomials orthonormal with respect to the weight function times the square root of the weight function. It turns out that in this case the skew-orthogonal polynomials are simply expressed in terms of the φ_j. In fact, in the notation of sec. 8, we have

$$\psi_{2n+1} = \varphi_{2n+1}/\sqrt{2}, \quad \psi_{2n} = -\varepsilon\varphi_{2n+1}/\sqrt{2}.$$

In other words,

$$p_{2n+1}(x) = e^{x^2/2}\,\varphi_{2n+1}(x)/\sqrt{2}, \quad p_{2n}(x) = -e^{x^2/2}\,\varepsilon\varphi_{2n+1}/\sqrt{2}.$$

The point is that p_{2n} as so defined is actually a polynomial of degree $2n$, as is easily seen. The skew-orthogonality is easy. If j and k are of the same parity the corresponding matrix entry equals 0, and if they have opposite parity we compute, (using the fact that $D\varepsilon =$ Identity)

$$\int (p_{2n}(x)\,p'_{2m+1}(x) - p'_{2n}(x)\,p_{2m+1}(x))\,e^{-x^2}\,dx = \int (\psi_{2n}\,\psi'_{2m+1} - \psi'_{2n}\,\psi_{2m+1})\,dx$$

$$= \frac{1}{2}\int (\varphi_{2n+1}\,\varphi_{2m+1} - \varepsilon\varphi_{2n+1}\,\varphi'_{2m+1})\,dx.$$

Integration by parts applied to the second integrand and the orthonormality of the φ_j show that the above equals $\delta_{n,m}$.

We now compute the entries of the matrix (8.1). If we keep in mind that the inverse of M in this case equals $-M$ we see that $S_N(x,y)$ is equal to $1/2$ times

$$\sum_{n=0}^{N-1} \varphi_{2n+1}(x)\,\varphi_{2n+1}(y) - \sum_{n=0}^{N-1} \varphi'_{2n+1}(x)\,\varepsilon\varphi_{2n+1}(y). \tag{10.1}$$

To compute the second sum we use the differentiation formula

$$\varphi'_i = \sqrt{\frac{i}{2}}\,\varphi_{i-1} - \sqrt{\frac{i+1}{2}}\,\varphi_{i+1}.$$

If (a_{ij}) is the antisymmetric tridiagonal matrix with $a_{i,\,i-1} = \sqrt{i/2}$ then

$$\varphi_i' = \sum_{j \geq 0} a_{ij}\, \varphi_j, \quad \varphi_j = \sum_{i \geq 0} a_{ji}\, \varepsilon\varphi_i. \tag{10.2}$$

The first is just a restatement of the differentiation formula and the second follows from the first by applying ε and interchanging i and j. By the first part of (10.2) the second sum in (10.1) equals $\sum a_{ij}\, \varphi_j \cdot \varepsilon\varphi_i$ summed over all $i, j \geq 0$ such that i is odd and $\leq 2N - 1$. Using the antisymmetry of (a_{ij}) and a different description of the range of summation, we see that this equals $-\sum \varphi_j \cdot a_{ji}\, \varepsilon\varphi_i$ summed over all $i, j \geq 0$ such that j is even and $\leq 2N$ except for the single term corresponding to $i = 2N + 1$, $j = 2N$. (Recall that $a_{ij} = 0$ unless $|i - j| = 1$.) By the second part of (10.2) this equals

$$-\sum_{n=0}^{N} \varphi_{2n}(x)\, \varphi_{2n}(y) - \sqrt{N + \tfrac{1}{2}}\, \varphi_{2N}(x)\, \varepsilon\varphi_{2N+1}(y). \tag{10.3}$$

Recall that $S_N(x, y)$ equals $1/2$ times the sum (10.1). If we now denote by $S_N(x, y)$ the sum itself, then

$$S_N(x, y) = \sum_{n=0}^{2N} \varphi_n(x)\, \varphi_n(y) + \sqrt{N + \tfrac{1}{2}}\, \varphi_{2N}(x)\, \varepsilon\varphi_{2N+1}(y),$$

and consequently

$$IS_N(x, y) = \sum_{n=0}^{2N} \varepsilon\varphi_n(x)\, \varphi_n(y) + \sqrt{N + \tfrac{1}{2}}\, \varepsilon\varphi_{2N}(x)\, \varepsilon\varphi_{2N+1}(y),$$

$$S_N D(x, y) = -\sum_{n=0}^{2N} \varphi_n(x)\, \varphi_n'(y) - \sqrt{N + \tfrac{1}{2}}\, \varphi_{2N}(x)\, \varphi_{2N+1}(y).$$

(That $IS_N(x, y)$ is obtained by applying ε to (10.1) as a function of x is a consequence of the fact that only odd indices occur there.) Hence our matrix kernel is given by

$$K_N(x, y) = \frac{1}{2} \begin{pmatrix} S_N(x, y) & S_N D(x, y) \\ \\ IS_N(x, y) & S_N(y, x) \end{pmatrix},$$

in analogy with (6.4).

The Gaussian orthogonal ensemble ($\beta = 1$). In order to continue using the same functions φ_j as before we shall now take as our weight function $w(x) = e^{-x^2/2}$. Thus the functions ψ_j of sec. 9 will be of the same form, polynomial times $e^{-x^2/2}$. For the p_j to be skew-orthogonal in the present context the antisymmetric matrix M with j, k entry

$$\int \int \varepsilon(x - y)\, \psi_j(x)\, \psi_k(y)\, dy\, dx = \int \psi_j(x)\, \varepsilon\psi_k(x)\, dx$$

must be a direct sum of $N/2$ copies of Z. (Recall that N is even.) The ψ_j are again very simple. They are given by

$$\psi_{2n} = \varphi_{2n}, \quad \psi_{2n+1} = \varphi_{2n}'.$$

20

That these give polynomials of the right degree (and parity) is clear, and we compute

$$\int \psi_{2n}(x)\, \varepsilon\psi_{2m+1}(x)\, dx = \int \varphi_{2n}(x)\, \varphi_{2m}(x)\, dx = \delta_{n,m}.$$

The sum $S_N(x,y)$ in (9.1) is now given by

$$\sum_{n=0}^{\frac{N}{2}-1} \left(\psi_{2n}(x)\, \varepsilon\psi_{2n+1}(y) - \psi_{2n+1}(x)\, \varepsilon\psi_{2n}(y)\right) = \sum_{n=0}^{\frac{N}{2}-1} \varphi_{2n}(x)\, \varphi_{2n}(y) - \sum_{n=0}^{\frac{N}{2}-1} \varphi'_{2n}(x)\, \varepsilon\varphi_{2n}(y).$$

By (10.2) the second sum on the right, equals $\sum a_{ij}\, \varphi_j \cdot \varepsilon\varphi_i$ summed over all $i, j \geq 0$ such that i is even and $\leq N-2$. This equals $-\sum \varphi_j \cdot a_{ji}\, \varepsilon\varphi_i$ summed over all $i, j \geq 0$ such that j is odd and $\leq N-1$ except for the single term corresponding to $i = N$, $j = N-1$. This sum equals

$$-\sum_{n=0}^{\frac{N}{2}} \varphi_{2n-1}(x)\, \varphi_{2n-1}(y) - \sqrt{\frac{N}{2}}\, \varphi_{N-1}(x)\, \varepsilon\varphi_N(y).$$

Hence we have now

$$S_N(x,y) = \sum_{n=0}^{N-1} \varphi_n(x)\, \varphi_n(y) + \sqrt{\frac{N}{2}}\, \varphi_{N-1}(x)\, \varepsilon\varphi_N(y),$$

and consequently the other entries of the matrix $K_N(x,y)$ in (9.1) are given by

$$IS_N(x,y) = \sum_{n=0}^{N-1} \varepsilon\varphi_n(x)\, \varphi_n(y) + \sqrt{\frac{N}{2}}\, \varepsilon\varphi_{N-1}(x)\, \varepsilon\varphi_N(y),$$

$$S_N D(x,y) = -\sum_{n=0}^{N-1} \varphi_n(x)\, \varphi'_n(y) - \sqrt{\frac{N}{2}}\, \varphi_{N-1}(x)\, \varphi_N(y).$$

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