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Final Report
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Number of Eigenstates of the Scattering Process

Under low temperature and low pressure, lights follow quantum mechanics and act as particles. When light or electronic wave omits, the particles will experience scattering processes. And that determines the electronic,thermal conductivity and many other transport properties. Therefore, it is crucial to figure out as much information of the particle as possible before and after this process. And since the initial states are controllable so the only problem is what happens after lights scatter.

Assume the wavefunction of the incoming particle can be represented by:
$\Phi in = \sum_{i=1} (ai * \varphi i)$, where ψi are the states of particle and ai is the coefficient or amplitudes, and the outgoing particle can be written as:
$\Phi out = \sum_{i=1} (bi * \varphi out)$.

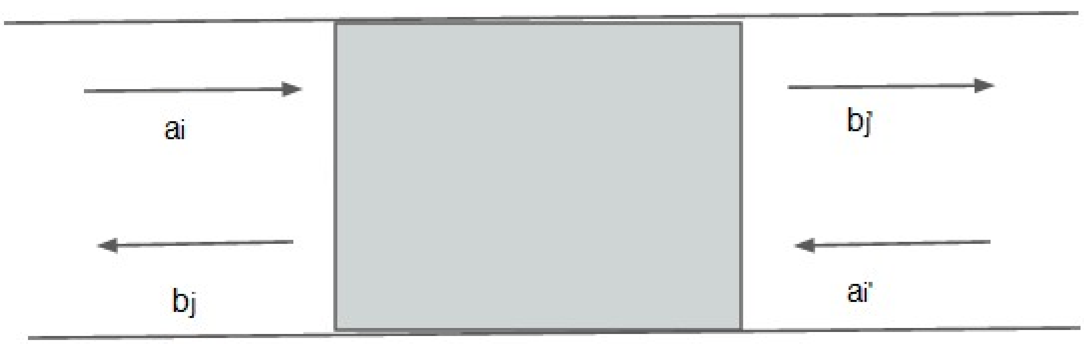

Figure 1, Scattering process. ai is the amplitude of incoming wave from the left, bj is the amplitude of the reflected wave from the left. Similarly, ai* and bj* have the same meaning except that they are from the right side.

The scattering matrix transfers the income into the outgoing wave function, as showed below:
$\Phi out = S * \Phi in$.
Depend on different models, the scattering (S) matrix will generate both or one of the real and complex eigenvalues. The complex numbers imply the phase transitions and are correlated with the collapse of wave functions, therefore troubling the simulation. Consequently, complex numbers are not welcomed. Real eigenvalues, instead, reveal a clear and neat picture of the states

for poping.So far, transport properties, suvh as conductivities, can then be obtained.

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Here the problem can be converted to how many real eigenvalues are there for a random matrix. Studied by Dr. A. Edelman in 1993, the expectation value of the number of real eigenvalues ($\overline{n}$) decays at the same rate as that of $\sqrt{k}$ where k is the size of matrix. And the following equation exists:

$$\lim_{n\to\infty}\frac{\overline{n}}{\sqrt{k}}=\sqrt{\frac{2}{\pi}}$$

After expanding, its asymptotic series is $\overline{n}=\sqrt{\frac{2k}{\pi}}\left(1-\frac{1}{4n}-\frac{1}{32n^2}-...\right)$.

Particularly, for even size of matrix, $\overline{n}=\sqrt{2}*\sum_{k=0}^{n/2-1}\left(\frac{(4k-1)!!}{4k!!}\right)$;

for odd size of matrix, $\overline{n}=1+\sqrt{2}*\sum_{k=1}^{(n-1)/2}\left(\frac{(4k-3)!!}{(4k-2)!!}\right)$.

Monte Carlo experiments have been conducted to verify the theory first. Then the standard deviation has also been run. The result is, as the size of the matrix becomes very large, the standard deviation of the number of real eigenvalues approaches to zero.

1000 matrices of size 1*1, 50*50, 100*100 and 200*200 were run.

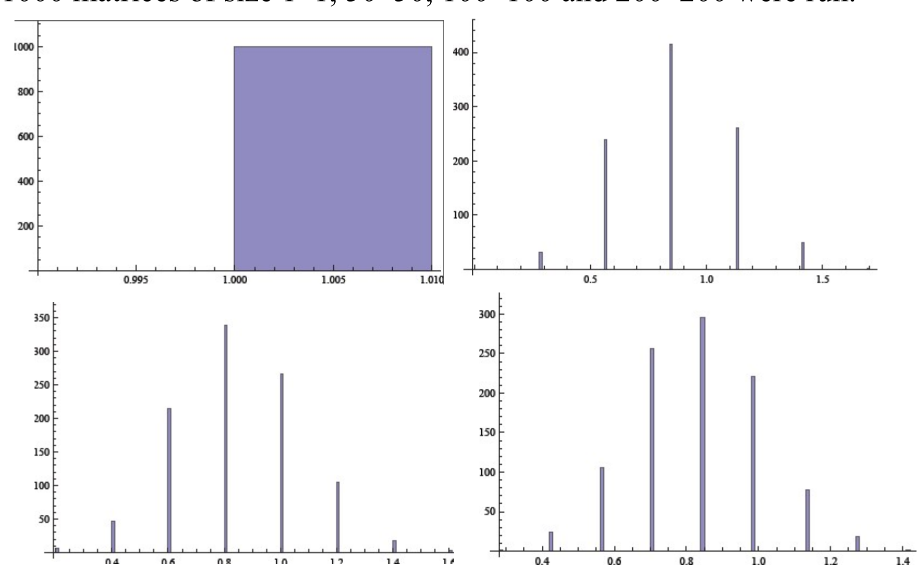

Figure , $\frac{n}{\sqrt{k}}$ of random matrix of size k*k: 1*1(Left top),50*50(right top),100*100(left bottom), 200*200(right bottom). n is the number of real eigenvalues of each matrices.

For 1*1 matrices, it is obvious that the number of real eigenvalue has to be 1. It is also ready to see that n has a few choices, n-2, n-4, n-6... That is because for any complex eigenvalues appearing, there must be the complex number itself and its conjugate number simultaneously.

Smooth out rest of the pictures, it is clear that all the lines are narrowing towards $0.8 \approx \sqrt{\frac{2}{\pi}}$.

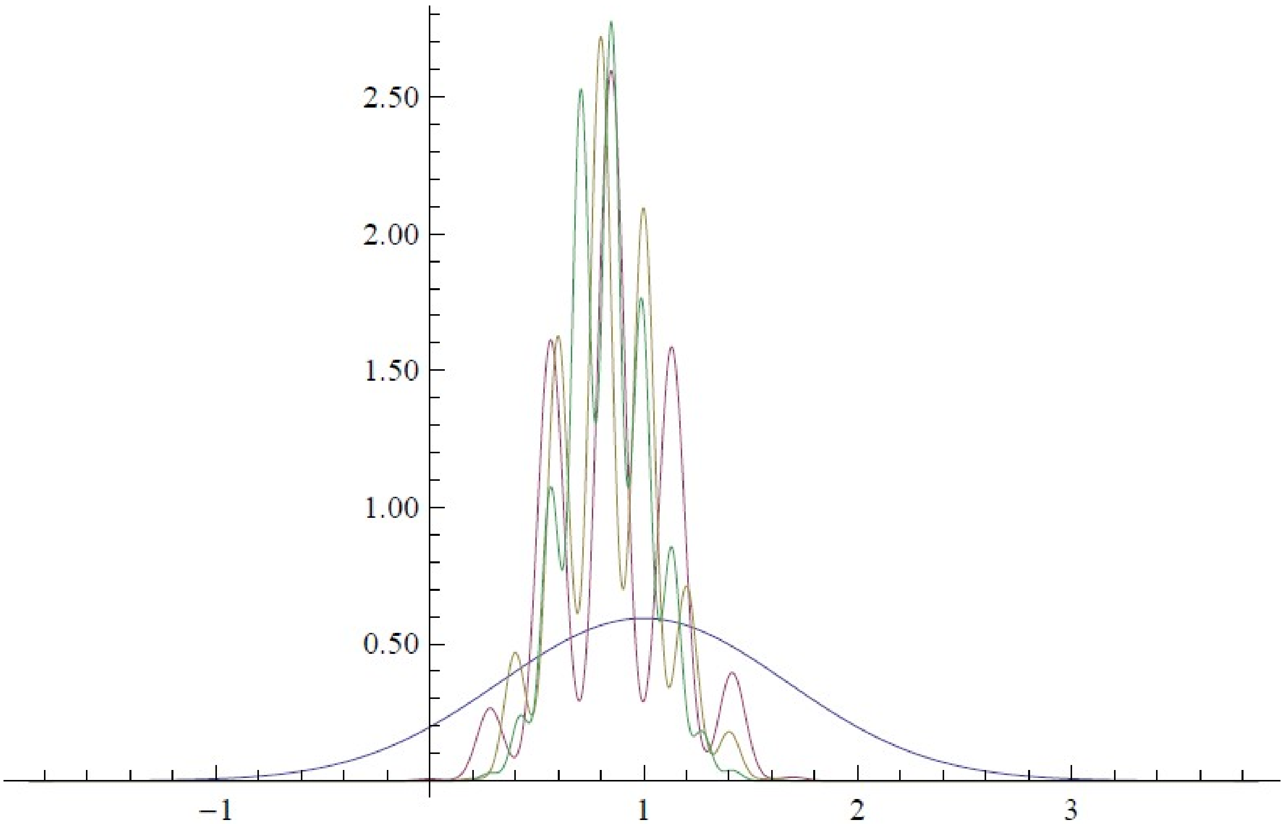

Figure 3, the blue curved lines represents for matrix of k=1. Other curves are for k= 50, 100 and 200, respectively.

Take the mean values of $\frac{n}{\sqrt{k}}$,

$\overline{\frac{n}{\sqrt{k}}} = \frac{\overline{n}}{\sqrt{k}}$.

For k=[1,n], Figure 4 is obtained. To see clearly the trend of the mean values, larger size of matrix from 100 to 200 was run. However, due to the larger amount of calculation time as k becomes larger, the experiment stopped at k=183. However, we could still see the mean has approached 0.80.

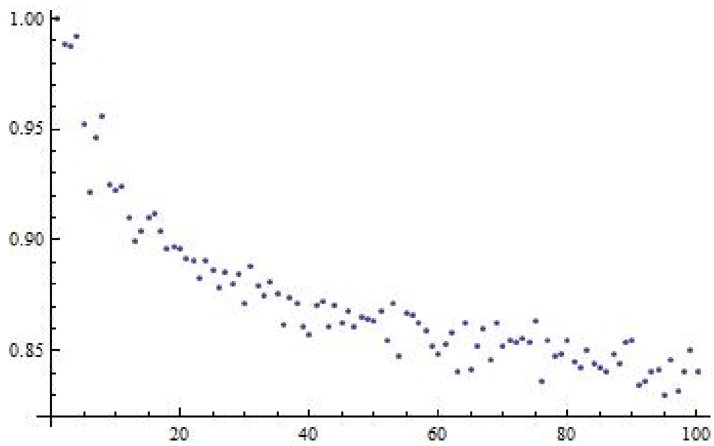

Figure 4, $\frac{\overline{n}}{\sqrt{k}}$ as k=[1,100].

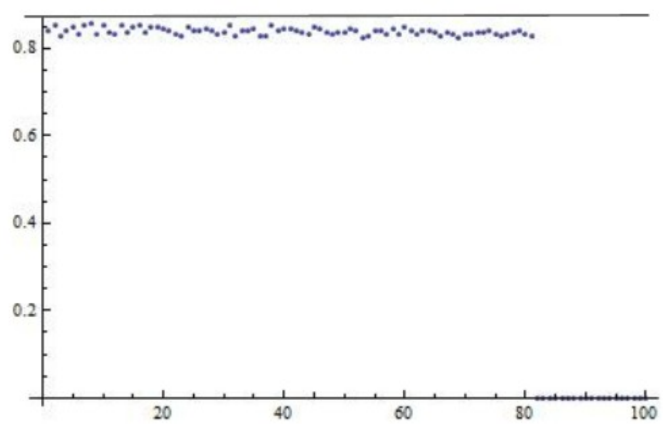

Figure 5, $\frac{\overline{n}}{\sqrt{k}}$ as k=[100, 183].

As implied by the previous figures, for large k, eventually all k by k matrices will have a special number of real eigenvalues, which is $0.8*\sqrt{k}$. Imaging for 1000000000 matrices with size of 1000000000*10000000000, the number of real eigenvalues should be 8*10^4 for each. On the other hand, very few matrices will have a different number from that. As a consequence, the standard deviation should goes to zero. Tested in the Monte Carlo experiment [see figure 6], it is clear to figure out. In this experiment, the number of trials dropped from 1000 to 100, since 1000 circular calculations will take incredible amount of time to run, especially for larger size matrix of 2000 by 2000 and up. With the number of trials decrease, only small size matrices are affected. The standard deviation of k=10 shift in between 0.41 to 0.38, while for large k, the shift is too small to count. Also, to make up for the declining trials, another experiment with 500 trials were done for k up to 3200 [see figure 7].

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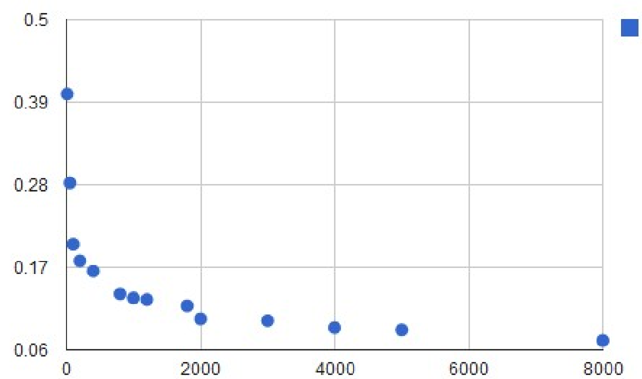


Figure 6, standard deviation of the number of real eigenvalues for matrix size of 10*10, 50*50, 100*100, 200*200, 400*400, 800*800, 1000*1000, 1200*1200, 1800*1800, 2000*2000, 3000*3000, 4000*4000, 5000*5000, 8000*8000. - 100 trials.

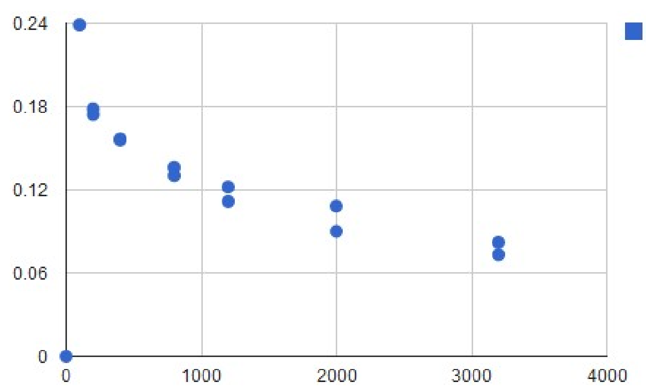
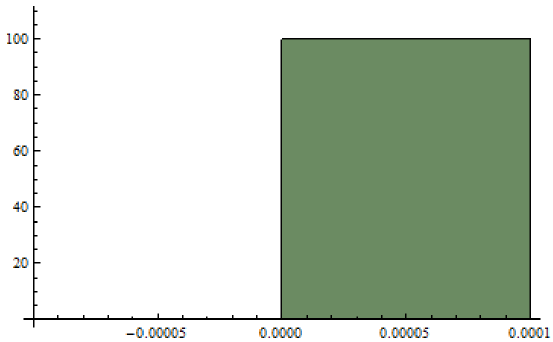


Figure 7, STD of the number of real eigenvalues for matrix size of k= 1, 100,101, 200, 201, 400, 401, 800,801,1200,1201,2000,2001,3200,3201. Two curves exist for even and odd k, respectively.-500 trials

Interesting but not surprised, for even and odd k, two curves appeared on their way to 0. The upper one is for even k while the lower one is for odd k. More experiments have been further run for unitary matrix and orthogonal matrix. Those special properties are added onto the scattering matrix in the Dyson model. And the results are not surprising.

Unitary matrices are designed as:
U= MatrixExp[i(M+transpose(M)/2)],* where M is a random matrix of k by k.
0 real eigenvalues can be observed. As a unititary matrix, all the eigenvalues spread around the unit circle on the complex plane. So it is reasonable that 0 real eigenvalues can be obtained.



For the orthogonal matrix, it is designed as
MatQ= QRDecompositon[M], where M is a random matrix. of k by k
For even k, the possible real eigenvalues are 1 and -1, which gives a total number of 2.
For odd k, the real eigenvalues is either 1 or -1, which give a total number of 1.

With 10 trials, table 1 is quickly generated. And the general value of $\frac{\overline{n}}{\sqrt{k}}$ can be written in:
$\frac{2}{\sqrt{k}}$, for k is even; $\frac{1}{\sqrt{k}}$, for k is odd.

10	**.632**
11	**.3015**
50	**.2828**
51	**.14**
100	**.2**
101	**.0995**

Table 1, for k= 10, 11 , 50, 51, 100, 101, the value of $\frac{\overline{n}}{\sqrt{k}}$ are showed on the right side of the table.-10trials

To improve the accuracy, more trials are tested and figure 9 is generated.

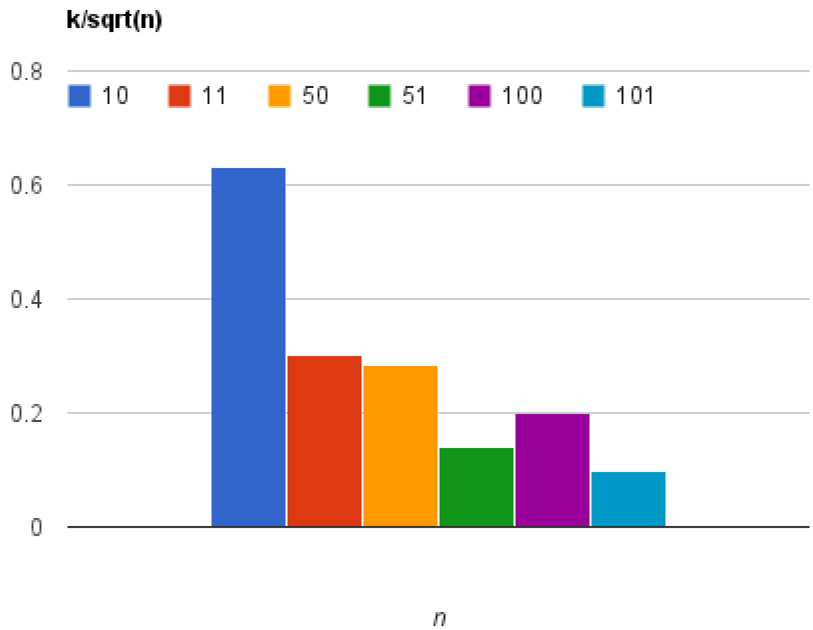


Figure 9, the average Vs. the size of the eigenvalues. Tested with 1000 trials. n in this figure is equivalent to k, the size of the matrix.

As showed in figure 9, table 1 can be applied to all sizes orthogonal matrices for all trials.

As a brief summary, states in the Dyson model will follow the unitary and the orthogonal patterns and therefore the scattering matrix generates a very few number of real eigenvalues with large k. In the other cases where no other restrictions will be applied to the S matrix, the number of states will be 0.8*$\sqrt{k}$. And since the standard deviation is 0 as k is large, the number of states for particles should have few fluctuation. Afterwards, the mean field can be introduced to calculate the other transport properties. And that will be the next step in the future research.

Last but not the least, as the initial requirements of this class, two tables for the probability of real eigenvalues as k=10 and k=11 are attached. Both calculations follow Dr. Edelman's method. When k=10, the probability of having 10, 8, 2 and 0 real eigenvalues can be obtained within 4 to 5 hours. However the rest two takes more than 10 hours. But it is still not hard to derive one of them by integrating y_i and xi. Eventually the leftover can be obtained by the fact that the total probability should be 1. In addition, experimental data are also collected.

k	$P_{10}(k)$	Monte Carlo results
10	$\frac{1}{(4193304\sqrt{2})}$	1.68*10^-7
8	$\frac{236539-320\sqrt{2}}{536870912\sqrt{2}}$	3.1*10^-4
6	$\frac{851546112-578296961\sqrt{2}}{536870912\sqrt{2}}$	0.0444
4	$1-\frac{1}{(4193304\sqrt{2})}-\frac{236539-320\sqrt{2}}{536870912\sqrt{2}}-\frac{851546112-578296961\sqrt{2}}{536870912\sqrt{2}}-\frac{1216831949-594932556\sqrt{2}}{536870912\sqrt{2}}-\frac{-1146637039+834100651\sqrt{2}}{536870912\sqrt{2}}$	0.421
2	$\frac{1216831949-594932556\sqrt{2}}{536870912\sqrt{2}}$	0.49
0	$\frac{-1146637039+834100651\sqrt{2}}{536870912\sqrt{2}}$	0.043

Table 2, Pk(n), the probability of n real eigenvalues for k by k matrices.

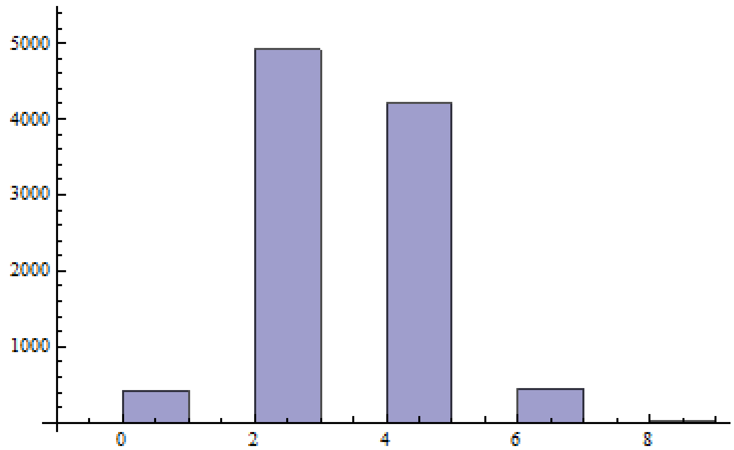


Figure 10, Monte Carlo experiment with 10000 trials.

k	$P_{11}(k)$	
11	$\frac{1}{134217728\sqrt{2}}$	5.27*10^-9
9	$\frac{-330+333123\sqrt{2}}{8589934592\sqrt{2}}$	3.87*10^-5
7	$\frac{-3.35*10^{15}+2.367*10^{15}\sqrt{2}}{8589934592\sqrt{2}}$	8.9*10^-3
5	$\frac{-2.32*10^{15}+2.439*10^{15}\sqrt{2}}{8589934592\sqrt{2}}$	0.2102
3	1- all other numbers	0.5818
1	$\frac{-12606311702+10629845251\sqrt{2}}{8589934592\sqrt{2}}$	0.1997

Table 3, Pk(n), the probability of n real eigenvalues for k by k matrices.

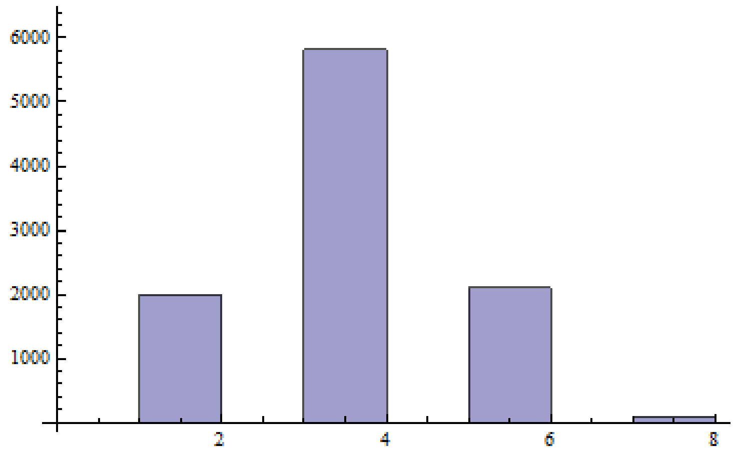


Figure 11, Monte Carlo experiment with 10000 trials.

Reference

Edelman A, Kostlan E,*How many Eigenvalues of a Random Matrix are Real*, July 3,1993

C. W. J. Beenakker, *Random-matrix theory of thermal conduction in superconducting quantum dotsRandom-matrix theory of thermal conduction in superconducting quantum dots.* Apr 2010

Michael V. Moskalets,*Scattering matrix approach to non-stationary quantum transport*