Random Matrix Theory and Non-coherent MIMO Communications

Mina Karzand

Massachusetts Institute of Technology

mkarzand@mit.edu

May 14, 2012

Classic Communications

- $x[t]$: for $t = 1, ..., T$ Input sequence containing information to be transmitted
- $y[t]$: for $t = 1, ..., T$ Output sequence to be decoded at the Rx
- $P(y|x)$: Stochastic model of communication channel

Main Questions

What is the maximum amount of information that can be transmitted *reliably* over a given channel? (Maximum rate: **Capacity**)
How to achieve the maximum rate? (Optimal Encoding and Decoding)

Point-to-Point Wireless Channel

$$y[t]=\text{SNR}\ h[t]x[t]+w[t], \qquad \text{for } t=1,...,T$$

- $w[t] \sim \mathcal{N}(0,1)$ Communication noise
- $h[t] \sim \mathcal{N}(0,1)$ Fading coefficient (due to multi-path and fading)
- SNR Signal to noise power ratio

Point-to-Point Wireless Channel

$$y[t] = \text{SNR}\ h[t]x[t] + w[t], \qquad \text{for } t = 1, ..., T$$

Input power constraint: $\frac{1}{T}\sum_{t=1}^{T} |x[t]|^2 < P$

What is troubling?

- noise (low SNR)
- deep fade ($|h[t]|^2 << 1$)

Coherent Point-to-Point Comm.

Rx know the channel coefficients Rx adds redundancy to input signal over time to embed information reliably.

$$C_{\text{coh}} = \mathbb{E}_{h^T}\left[\frac{1}{T}\max_{P_{X^T}} I(X^T; Y^T)\right]$$

$SNR|h[t]|^2$ is a measure of the quality of channel at time t

- Tx invests larger input signal power when $SNR|h[t]|^2$ is small
- Tx invests smaller input signal power when $SNR|h[t]|^2$ is large

Non-coherent Point-to-Point Comm.

- Rx does not know the values of channel coefficients
- Tx should design the input signal only using the statistics of h^T

e.g., Amplitude Modulation $x[t] \in \{0, A\}$

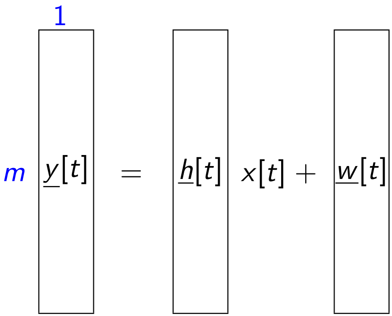
Flat Fading

$h[t] = h, \qquad \text{for } t = 1, ..., T$

SIMO

m Rx antennas

$$y_j[t] = h_j[t]x[t] + w_j[t] \qquad \text{for } t = 1, ..., T, j = 1, ..., m$$
$$\underline{y}[t] = \underline{h}[t]x[t] + \underline{w}[t]$$



MISO

$$y[t] = \sum_{i=1}^{n} h_i[t]x_i[t] + w[t] \qquad \text{for } t = 1, ..., T, i = 1, ..., n$$

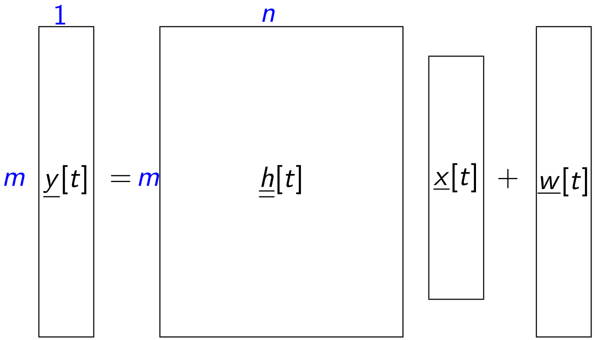
$$y[t] = \underline{h}'[t].\underline{x}[t] + w[t]$$

$$y[t] = \underbrace{\overbrace{\boxed{\underline{h}'[t]}}^{n}}_{1} \; \boxed{\underline{x}[t]} + w[t]$$

MIMO

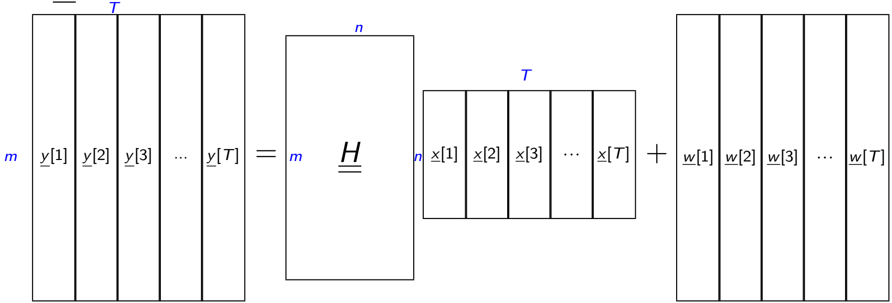
$$y_j[t] = \sum_{i=1}^{n} h_{ij}[t] x_i[t] + w_j[t] \qquad \text{for } t = 1, \ldots, T \qquad j = 1, \ldots, m$$

$$i = 1, \ldots, n$$



Flat fading MIMO

$\underline{\underline{h}}[t] = \underline{\underline{H}}$



$$\underline{\underline{Y}}_{m\times T} = \underline{\underline{H}}_{m\times n} \cdot \underline{\underline{X}}_{n\times T} + \underline{\underline{W}}_{m\times T}$$

Degrees of freedom

$$\underline{\underline{Y}}_{m\times T} = \underline{\underline{H}}_{m\times n}.\underline{\underline{X}}_{n\times T} + \underline{\underline{W}}_{m\times T}$$

$$\underline{\underline{Y}} \in \mathcal{C}^{m\times T}$$

- high SNR
- *DOF=d*: How many dimensions of $\underline{\underline{Y}}$ is only affected by $\underline{\underline{X}}$?
- *Pre-log factor*:

$$C \sim d.\log(SNR)$$

The optimal input distribution:

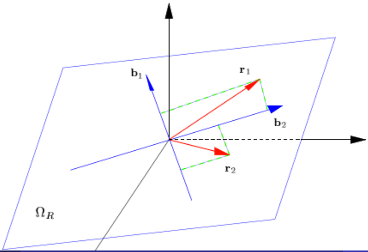
$$X = A\Theta$$

- Θ : $M \times T$ isotropically distributed unitary matrix. (Its i-th row represents the direction of the transmitted signal from ith antenna)
- A : $M \times M$ real diagonal matrix representing norm of the signal (average input power constraint)
- Θ independent of A

Change of Coordinates

$$\begin{aligned} \mathcal{C}^{M\times T} &\rightarrow \mathcal{C}^{M\times M} \times G(T,M) \\ R &\rightarrow (C_R, \Omega_R) \end{aligned}$$

- R: $M \times T$ matrix with $T \geq M$.
- Ω_R: subspace spanned by the rows of R ($\Omega_R \in G(T,M)$)
- $G(T,M)$ *Grassman Manifold:* the set of all M dimensional subspaces of $\mathcal{C}^T$
- C_R: the M row vectors of R with respect to a canonical basis in Ω_R



Change of Coordinates

$$\begin{aligned} \mathcal{C}^{M\times T} &\rightarrow \mathcal{C}^{M\times M} \times G(T,M) \\ R &\rightarrow (C_R, \Omega_R) \end{aligned}$$

Not considering the additive noise, $Y_0 = HX$

$$\Omega_{HX} = \Omega_X \qquad \text{with probability 1}$$

The fading coefficients H affect X by changing C_X and leave Ω_X unchanged

- Ω_X is corrupted only by noise
- C_X is corrupted both by noise and channel fading

$$\begin{aligned} I(X;Y) &= I(\Omega_X;Y) + I(C_X;Y|\Omega_X) \\ &= I(\Omega_X;Y) \end{aligned}$$

Change of Coordinates

$$\begin{array}{rcl} \mathcal{C}^{M\times T} & \rightarrow & \mathcal{C}^{M\times M} \times G(T,M) \\ R & \rightarrow & (C_R, \Omega_R) \end{array}$$

Theorem

If $R \in \mathcal{C}^{M\times T}$ is isotropically distributed (i.e., $\forall Q, p(R) = p(RQ)$), then

$$h(R) = h(C_R) + \log|G(T,M)| + (T-M)\mathbb{E}[\log\det RR']$$

- $h(C_R) + \log|G(T,M)|$ *differential entropy of R in $\mathcal{C}^{M\times M} \times G(T,M)$.*
- $(T-M)\mathbb{E}[\log\det RR']$ *Jacobian term for the coordinate change.*

Capacity in high SNR regime

$$I(X;Y) = h(Y) - h(Y|X)$$

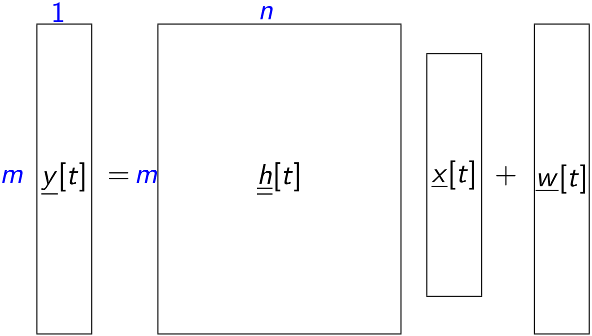
$$\begin{aligned} h(Y|X) &= m\mathbb{E}[\log\det A^2] + m^2\log(\pi e) + m(T-m)\log(\pi e\sigma^2) \\ h(Y) &\approx h(HX) \\ &= h(C_{HX}) + \log|G(T,M)| + (T-m)\mathbb{E}[\log\det HA^2H'] \\ &= h(C_{HX}) + \log|G(T,M)| + (T-m)\mathbb{E}[\log\det HH' + \log\det A^2] \end{aligned}$$

Optimal power allocation is const. for all antennas Information is only conveyed through Ω_X

Non-flat fading MIMO

$$y_j[t] = \sum_{i=1}^{n} h_{ij}[t]x_i[t] + w_j[t] \qquad \text{for } t = 1, \ldots, T \qquad j = 1, \ldots, m$$

$$i = 1, \ldots, n$$



Non-flat fading MIMO

For all i, j

$$\mathbb{E}\left(\begin{bmatrix} h_{ij}(1) \\ h_{ij}(2) \\ \vdots \\ h_{ij}(T) \end{bmatrix} [h_{ij}(1), h_{ij}(2), \cdots, h_{ij}(T)]\right) = K_H$$

$$rank(K_H) = Q << T$$

- What is the degrees of freedom of the system?
- What is the optimal signal distribution?