(1) 3/1/2013

Discrete Determinantal Point Process

Warm-up ($q=1$)

hence positive semi-def

Y .. tall skinny orthogonal matrix
$n \times p$, $Y^T Y = I_p$ $K = YY^T$ projection with eigs $1^p, 0^{n-p}$

$\operatorname{Tr} K = p$

Cauchy Binet $\det I_p = I_p\begin{pmatrix}1 \cdots p \\ 1 \cdots p\end{pmatrix} = \sum_{i_1 < \cdots < i_p} Y\begin{pmatrix}i_1 \cdots i_p \\ 1 \cdots p\end{pmatrix}^2 = 1$

$$K\begin{pmatrix}i_1 \cdots i_p \\ i_1 \cdots i_p\end{pmatrix} = Y\begin{pmatrix}i_1 \cdots i_p \\ 1 \cdots p\end{pmatrix}^2$$

Two Discrete Probabilities

a) Prob on $\{1, \dots, n\}$
$\Pr(i) = \frac{1}{p} K_{ii} = \frac{1}{p} \sum_{j=1}^{p} Y_{ij}^2$

b) Prob on p element subsets of $\{1, \dots, n\}$

$$\Pr(i_1, \dots, i_p) = Y\begin{pmatrix}i_1 \cdots i_p \\ 1 \cdots p\end{pmatrix}^2 = K\begin{pmatrix}i_1 \cdots i_p \\ i_1 \cdots i_p\end{pmatrix}$$

= det of principal submatrix of K obtained by taking rows & cols in $\{i_1, \dots, i_p\}$

What is the connection?
Answer: Take a random number uniformly from $i_1, \dots, i_p$ generated from the second and obtain the first.

② 3/1/2013

Proof:

$$\mathrm{Det}\left(Y^T\left(I+\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n\end{pmatrix}\right)Y\right) = 1 + \sum_{i\in\{i_1,\dots,i_n\}} z_i \, Y\begin{pmatrix} i_1 \cdots i_p \\ 1 \cdots p\end{pmatrix}^2 + \cdots$$

e.g. coef of z_1 $\displaystyle\sum_{i\in\{i_1,\dots,i_p\}} Y\begin{pmatrix} i_1 \cdots i_p \\ 1 \cdots p\end{pmatrix}^2 = \Pr(i \text{ shows up})$

$$\Pr(i) = \frac{1}{p}\sum_{i\in\{i_1,\dots,i_p\}} Y\begin{pmatrix} i_1 \cdots i_p \\ 1 \cdots p\end{pmatrix}^2 = \Pr(i \text{ is chosen})$$

$$\mathrm{Det}\left(I + K\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n\end{pmatrix}\right) = 1 + \sum z_i K_{ii} + \cdots \qquad (1)$$

$$= \sum_{\Delta\subset\{1,\dots,n\}} z_\Delta K\begin{pmatrix}\Delta \\ \Delta\end{pmatrix}$$

can omit but larger are 0 → $(|\Delta| \le p)$

Thus $\Pr(i) = \frac{1}{p} K_{ii}$ as defined in (a)

z_Δ takes column indices from Δ in $K\begin{pmatrix} z_1 & \\ & \ddots \end{pmatrix}$ and the rest from I

③

The bigger story

$1 \le q \le p$ (q fixed)

$\{i_1, \dots, i_p\}$ generated with prob $K\begin{pmatrix} i_1 \dots i_p \\ i_1 \dots i_p \end{pmatrix}$ as before

Now pick q at random, call it $\mathcal{J}_q \subset \{1, \dots, n\}$ of size q, uniformly ($Pr = 1/\binom{p}{q}$)

<u>Theorem</u>: $Pr(\mathcal{J}_q) = \frac{1}{\binom{p}{q}} K\begin{pmatrix} \mathcal{J}_q \\ \mathcal{J}_q \end{pmatrix}$

Need to show $\sum_{\mathcal{J}_q \subset \{1, \dots, p\}} K\begin{pmatrix} i_1 \dots i_p \\ i_1 \dots i_p \end{pmatrix} = K\begin{pmatrix} \mathcal{J}_q \\ \mathcal{J}_q \end{pmatrix}$

since $\frac{1}{\binom{p}{q}} \sum_{\mathcal{J}_q \in \{1, p\}} K\begin{pmatrix} i_1 \dots i_p \\ i_1 \dots i_p \end{pmatrix} =$ the prob that $\mathcal{J}_q$ is picked at random from $i_1 \dots i_p$

but obviously coef of expansion (1) of $Z_{\mathcal{J}_q}$

Focus on pairs $Pr(\{i, j\}) = \frac{1}{\binom{p}{2}} \left[K_{ii} K_{jj} - K_{ij}^2 \right]$

$Pr(i, j) = \frac{1}{p(p-1)} \left[K_{ii} K_{jj} - K_{ij}^2 \right] \approx \frac{1}{p^2} \left[K_{ii} K_{jj} - K_{ij}^2 \right]$
ordered

K_{ij} is like a $1 - \frac{K_{ij}^2}{K_{ii} K_{jj}}$ is like a correlation

4

$n \to$ continuous

$Y = [\phi_0 \ldots \phi_{p-1}]$ where $\int_{\text{interval}} \phi_i(x)\phi_j(x)\,dx = \delta_{ij}$

$Y^T Y = I_p$

$YY^T = K \quad K(x,y) = \sum_{i=0}^{p-1} \phi_i(x)\phi_i(y)$ ← rank p

$K(x,x) = \frac{1}{p}\sum_{i=0}^{p-1} \phi_i^2(x)$ is a prob density

$$\int_{x_1<\ldots<x_p}\!\!\int K\begin{pmatrix} x_1 \ldots x_p \\ x_1 \ldots x_p \end{pmatrix} dx_1 \ldots dx_p = 1$$

or

$$\frac{1}{p!}\int \ldots \int_{\mathbb{R}^p} K\begin{pmatrix} x_1 \ldots x_p \\ x_1 \ldots x_p \end{pmatrix} dx_1 \ldots dx_p = 1$$

Two are related

Pick $\{x_1, \ldots, x_p\}$ randomly
then pick one uniformly
or a subset uniformly

5

Special Case of orthogonal polynomial ensembles

Start with weight $w(x)$ ~~[illegible]~~ on $\mathbb{R}$ (or subinterval)

$$\phi_i(x) = \pi_i(x)\sqrt{w(x)} = \left(c\,\pi_i(x)\sqrt{\tfrac{w(x)}{c^2}} \right)$$

π_i: orthogonal polynomial of degree i

so arbitrary constant in weight

Sometimes nice to make it a prob measure

$$K\begin{pmatrix} x_1 \ldots x_p \\ x_1 \ldots x_p \end{pmatrix} = \begin{vmatrix} \phi_0(x_1) & \cdots & \phi_{p-1}(x_1) \\ \vdots & & \vdots \\ \phi_0(x_p) & \cdots & \phi_{p-1}(x_p) \end{vmatrix}^2$$

$$= \prod_{i=0}^{p-1} (\text{leading coef of } \pi_i) \prod |x_i - x_j|^2 \prod_{i=1}^{p} w(x_i)$$

$$\begin{pmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{p-1} \\ & & b_{p-1} & a_p \end{pmatrix} \begin{pmatrix} \pi_0 \\ \vdots \\ \pi_p \end{pmatrix} = x \begin{pmatrix} \pi_0 \\ \vdots \\ \pi_p \end{pmatrix} - b_p \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pi_{p+1} \end{pmatrix}$$

$$\prod_{i=0}^{p-1} (\text{leading coef}) = b_1^{p-1} b_2^{p-2} \cdots b_{p-1}^{1}$$

Also

$$\mathrm{Det}\left(Y^T \left(I + \begin{smallmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{smallmatrix}\right) Y \right) = \mathrm{Det}\left(I + K\left(\begin{smallmatrix} z_1 & & \\ & \ddots & \\ & & z \end{smallmatrix}\right)\right)$$

$$= \sum \prod_{i \in \{i_1, \dots, i_p\}} (1+z_i) \; Y\begin{pmatrix} i_1 & \dots & i_p \\ 1 & \dots & r \end{pmatrix}^2$$

$$= E\left(\prod (1+z_i) \right)$$

e.g. Take $z_1 = \dots = z_p = 0$ rest -1

get Prob all $< r$

$$= \mathrm{Det}(I - K|_{\geq r})$$

Continuous $\mathrm{Det}\left(I - \left[\int_{x \geq t} \phi_i(x) \phi_j(x)\, dx \right] \right) = Pr[\text{all } \lambda_i \leq t]$

$$= F_{\lambda_{max}}(t)$$

$$\mathrm{Det}(I - K \chi_{x \geq t}) = F_{\lambda_{max}}(t)$$

$K\chi_{\geq t}$ is an operator that takes

$$f(y) \to \int_t^{\infty} \left(\sum \phi_i(x) \phi_i(y) \right) f(y)\, dy$$