3/13/2013

1

Sym Tridiagonal Matrices

$$T = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

Classical: $w(x) \longrightarrow \{\pi_j(x)\} \longrightarrow$ three term recurrence

Useful: Start with T

Classical: Continuous weight $w(x)$

For $n = 0, 1, 2, \ldots$

$$\begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ & & \beta_{n-1} & \alpha_n \end{pmatrix} \begin{pmatrix} \pi_0(x) \\ \vdots \\ \pi_{n-1}(x) \end{pmatrix} = x \begin{pmatrix} \pi_0(x) \\ \vdots \\ \pi_{n-1}(x) \end{pmatrix} - \beta_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pi_n(x) \end{pmatrix}$$

$$\pi_n(x) = \frac{(\beta_1 \cdots \beta_n)}{\sqrt{\int_{\mathbb{R}} w(x)\,dx}} \prod_{i=1}^{n} (x - \lambda_i)$$

→ eigenvalues of sym tridiagonals are fast + often very accurate

Note $$(T - xI)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = c \begin{pmatrix} \pi_0(x) \\ \vdots \\ \pi_{n-1}(x) \end{pmatrix}$$ if

x is not an eigenvalue

Since we know $\pi_0(x) = \dfrac{1}{\sqrt{\int w(x)\,dx}}$

we can compute the entire vector

$$\begin{pmatrix} \pi_0(x) \\ \vdots \\ \pi_{n-1}(x) \end{pmatrix} \qquad T = Q \Lambda Q^T$$

with a solve.

Can do spectrally

$$\sum Q(j,i) \frac{1}{\lambda_i - x} \cancel{Q^T e_n}\, Q(n,i)$$

$$\sum \frac{q_{n,i}}{\lambda_i - x} q_i$$

I think faster & more accurate than other ways.

For a discrete weight function say

$$w(x) = \sum_{i=1}^{n} q_i^2 \, \delta(x - \lambda_i) \qquad \left[\sum_{i} q_i^2 = 1\right] \qquad q_i \neq 0$$

Can only get finite tridiagonal

Lanczos construction:

Unstable:

$$K = [\, q \;\; \Lambda q \;\; \ldots \;\; \Lambda^{n-1} q \,]$$

$$QR = K$$

$$T = Q \Lambda Q^T$$

Stable:

$v_0 = 0, \quad v_1 = q$

$\beta_0 = 0$

For $j = 1 : n$

$\quad w_j \leftarrow \Lambda v_j$

$\quad \alpha_j \leftarrow (w_j, v_j)$ stops here when $j = n$

$\quad w_j \leftarrow w_j - \alpha_j v_j - \beta_j v_{j-1}$

$\quad \beta_{j+1} \leftarrow \|w_j\|$

$\quad v_{j+1} \leftarrow w_j / \beta_{j+1}$

end

Comment: Lanczos works for any Λ not just discrete

Tridiagonalizes a matrix

Generalizes Householder-reduction

Comment.

In discrete case $\pi_0, \dots, \pi_{n-1}$ well defined

π_n normalization not available

$$\sum_{k=1}^{n} \pi_i(\lambda_k)\,\pi_j(\lambda_k)\, q_k^2 = \delta_{ij}$$

$0 \le i, j \le n-1$

Gauss-Quadrature

$w(x)$ & $w_n(x)$ agree on polynomials up to degree $2n-1$

Continuous Lanczos

$$\Lambda = x$$

$$K = \sqrt{w(x)}\,[\,1 \;\; x \;\; x^2 \; \dots \; x^{n-1}\,] = QR$$

$$Q^* K Q^T = T$$

Equilibrium Measure

$$\rho(x) = \sum_{k=1}^{n} \frac{1}{n} \delta(x - \lambda_k)$$

$$\sum q_i^2 \delta(x - \lambda_k) \to w(x)$$

$$\sum \delta(x - \lambda_k) \to \rho(x)$$

R & R^{-1} transfer to monomial bases

$$\int \frac{\pi_j(t)\, w(t)}{x-t}\, dt \qquad \frac{1}{x-t} = \frac{1/x}{1-t/x}$$

$$= \int \left(\frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3} + \dots\right) \pi_j(t)\, w(t)\, dt$$

$$= \sum_{i=1} \frac{\text{coeff of } \cancel{t} \text{ in } \underbrace{\pi_j(t) \text{ when } t^i \text{ is expand}}_{R_{ji}}}{x^i}$$

1 ~~3/4~~ 2/27 – 3/6

Orthogonal Polynomial Eigenvalue Density

$w(x) \geq 0$ on I

$$\int_I \pi_i(x)\pi_j(x)w(x)\,dx = \delta_{ij}$$

Defn: We call the density

$$p(t_1,\dots,t_n) = c\,|\Delta|^\beta \prod_{i=1}^{n} w(t_i)$$

an orthogonal Polynomial (joint) Eigenvalue Density

when $\beta = 2$, this is called a determinantal point process or simply a determinantal process

$$p(t_1,\dots,t_n) = \det\big(K(t_i,t_j)\big)_{1\le i,j\le n} \quad \text{where } K(x,y) = \sum_{i=0}^{n-1} \pi_i(x)\pi_i(y)\,w(x)$$

~~where~~ Let $K = A^TA$ "Gram Matrix"

$$A = \begin{bmatrix} \pi_0(t_1) & \dots & \pi_0(t_n) \\ \vdots & & \\ \pi_{n-1}(t_1) & & \pi_{n-1}(t_n) \end{bmatrix} \begin{bmatrix} w(t_1)^{1/2} & & \\ & \ddots & \\ & & w(t_n)^{1/2} \end{bmatrix}$$

$$p(t_1,\dots,t_n) = \det(K) = \det(A)^2 = c\,(\text{Vandermonde})^2 \cdot \prod w(t_i)$$

$$Y^TY = I_p$$

We say that A is an Orthogonal Polynomial Model if its eigenvalues have the corresponding joint density.

Prefered are explicit element related formulas.
Of course one can take $x \in \mathbb{R}^n$ with an orthogonal polynomial density + $\mathrm{diag}(x)$ or $Q^T \mathrm{diag}(x) Q$ are models, for any distribution on Q, like Haar measure, but that seems like cheating.

Cauchy Binet

$C = AB$

$$C\begin{pmatrix} i_1 \cdots i_p \\ k_1 \cdots k_p \end{pmatrix} = \sum_{j_1 < j_2 < \cdots < j_p} A\begin{pmatrix} i_1 \cdots i_p \\ j_1 \cdots j_p \end{pmatrix} B\begin{pmatrix} j_1 \cdots j_p \\ k_1 \cdots k_p \end{pmatrix}$$

$$\det(A^T D A) = \sum A\begin{pmatrix} i_1 \cdots i_p \\ 1 \cdots p \end{pmatrix}^2 d_{i_1} d_{i_2} \cdots d_{i_p}$$

A is $n \times p$ $n \geq p$

$$D = I \qquad Pl(A) = \begin{pmatrix} \\ A\begin{pmatrix} i_1 \cdots i_p \\ 1 \cdots p \end{pmatrix} \\ \\ \end{pmatrix} \in \mathbb{R}^{\binom{n}{p}}$$

$\det(A^T A) = \|Pl(A)\|^2$ = volume squared of ~~parallelotope~~ $\{Ax : 0 \leq x \leq 1\}$

Generalized Euclidean formula:

$$\text{volume}^2 = \sum_{\binom{n}{p}} (\text{projection of volume to all coordinate ~~planes~~})^2$$

(3)

$$d_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

$$\operatorname{Det}(A^T D A) = \sum_{\substack{i_1 < \cdots < i_p \\ \text{all in } A}} A\binom{i_1 \cdots i_p}{1 \cdots p}^2$$

Let $Y^T Y = I$ "Tall skinny orthogonal"

span is a unit cube

$$d_i = \begin{cases} 1+z & \text{if } i = s \\ 1 & i \neq s \end{cases}$$

$$Y^T D Y = 1 + z \sum_{\substack{s \in \{i_1, \ldots, i_p\} \\ i_1 < i_2 < \cdots < i_p}} A\binom{i_1 \cdots i_p}{1 \cdots p}^2$$

$d_i = z_i$ symbolic

$$\det(A^T D A) = \sum A\binom{i_1 \cdots i_p}{1 \cdots p}^2 z_{i_1} \cdots z_{i_p}$$

$d_i = 1 + z_i$ symbolic

$$\sum_{k=0}^{p} \sum_{\substack{\{i_1, i_2, \ldots, i_k\} \subseteq \\ \{i_1, \ldots, i_p\} \\ i_1 < i_2 < \cdots < i_p}} \prod_{j=1}^{k} z_{i_j} \; A\binom{i_1 \cdots i_p}{1 \cdots p}^2$$

(4)

$$= \sum A\begin{pmatrix} i_1 \cdots i_p \\ 1 \cdots p \end{pmatrix}^2$$

$$+ \sum_{j \in \{i_1, \ldots, i_p\}} z_j \, A\begin{pmatrix} i_1 \cdots i_p \\ 1 \cdots p \end{pmatrix}^2$$

$$+ \sum_{\substack{\{j_1, j_2\} \subset \{i_1, \ldots, i_p\} \\ j_1 < j_2}} z_{j_1} z_{j_2} \; A\begin{pmatrix} i_1 \cdots i_p \\ 1 \cdots p \end{pmatrix}^2$$

$$\vdots$$

$$\sum_{i_1 < i_2 < \cdots < i_p} z_{i_1} z_{i_2} \cdots z_{i_p} \; A\begin{pmatrix} i_1 \cdots i_p \\ 1 \cdots p \end{pmatrix}^2$$

$= \cancel{\det(A^T(I+}$

If $A = Y$ (i.e. $Y^T Y = I_p$) then & $K = YY^T$

$$\det(Y^T(I + \mathrm{diag}(z_i))Y) = \det(I_p + Y^T \mathrm{diag}(z_i) Y)$$
$$= \det(I_n + K\,\mathrm{diag}(z_i))$$

Let $f = \mathrm{diag}(z_i)$ & we get

$\det(I + Kf)$

$$\Psi = \begin{pmatrix} \phi_0 & \cdots & \phi_{p-1} \end{pmatrix} \quad S \quad \Psi\begin{pmatrix} \cancel{t_1} \cdots \cancel{t_p} \\ 1 \cdots p \end{pmatrix}^2 = \rho(t_1, \ldots, t_p)$$

$$D = I + \delta(x-y)$$

$$\det(\Psi^T D \Psi) = 1 + \int_{\mathrm{sum}(x_i) = y} \rho(t_1, \ldots, t_p)\, dt_1 \ldots dt_p$$

$$(\Psi^T D \Psi)_{ij} = \delta_{ij} + \int \phi_i(x)\phi_j(\cancel{x})\,\delta(x-y)\,dx$$

$$= \delta_{ij} + \phi_i(y)\phi_j(y)$$

$$1 + \sum \phi_i^2(x) = 1 + \mathrm{Prob}(\mathrm{sum}\ x_i = y)$$

$D \Rightarrow I$

$D(x,x) = 1 + f(x)$

$$\det(\Psi^T(I+f)\Psi) = \underset{i=1}{\overset{p}{E}}\left(\left(\Pi\, f(t_i)\right)\right) \cancel{\partial x}$$

$$\det(\Psi^T\Psi + \Psi^T f \Psi)$$

$$= \det(I + \cancel{M}\,f_p) = \int \cdots \int \Pi(1+f(t_i)) = E(\Pi(1+f))$$

$$K_p = \Psi\Psi^T \quad K_p(x,y) = \sum_{i=0}^{n-1} \phi_i(x)\phi_i(y)$$

(6)

$$f(x) = \sum_{r=1}^{n} z_r \, \delta(x - y_r)$$

$= 0$ or z_r spike at y_r

analog to sum z's on

$$1 + \sum z_i \, M_1(y_i, y_i)$$

$$+ \sum z_i z_j \, M_2(y_i, y_j; y_i, y_j)$$

$\vdots$

$P(\text{all eigenvalues} < t)$

In general

$$\det(I+Kf) = E\left(\prod\left[1+f(x_n)\right]\right)$$

Remark: All can be done continuously

$f(x)$ can be $\sum_{r=1}^{p} z_r\,\delta(x-y_r)$ to set p x's to z

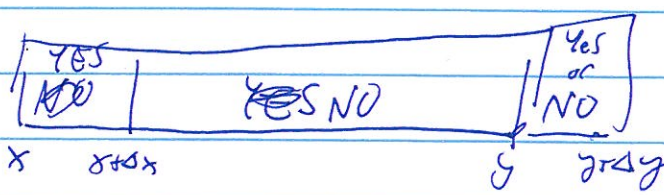
J can be any interval

$$\det(I-K\chi_J) = \Pr(\text{no eig in } J)$$

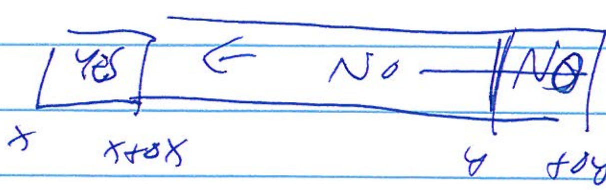
$$= \det\left[I - \left[\int_{x\in J}\phi_i(x)\,\phi_j(x)\,dx\right]\right]$$

$E(x,y) = \Pr$ no eig in $J=(x,y)$

$$\mathrm{I}\left(E(x+\Delta x, y) - E(x,y)\right) =$$



$$\mathrm{II}\left(E(x+\Delta x, y+\Delta y) - E(x, y+\Delta y)\right)$$



$$\mathrm{I} - \mathrm{II} =$$