18.338 Eigenvalues of Random Matrices

Problem Set 3

Due Date: Wed March 20, 2013

Reading and Notes

Read chapter *16* and *17* of the class notes. Please again give your feedback especially high level style and where things did not make sense, in addition to spelling or technical errors

Homework

Do the following three problems.

1. (M) The Christoffel-Darboux formula (also see equation (5.4) on page 59) states

$$\sum_{j=0}^{n} \pi_i(x)\pi_j(y) = \frac{k_n}{k_{n+1}} \frac{\begin{vmatrix} \pi_n(x) & \pi_n(y) \\ \pi_{n+1}(x) & \pi_{n+1}(y) \end{vmatrix}}{y - x},$$

where k_n is the lead coefficient of π_n. Let

$$\pi_j(x) = \frac{H_j(x)}{(\sqrt{\pi} j! 2^j)^{1/2}}$$

for Hermite Polynomials, then $k_n/k_{n+1} = \sqrt{n}$.

Use the known asymptotics

$$\lim_{m\to\infty} (-1)^m m^{1/4}\ \pi_{2m}(x)e^{-x^2/2} = \frac{\cos(\xi)}{\sqrt{\pi}}$$

$$\lim_{m\to\infty} (-1)^m m^{1/4}\ \pi_{2m+1}(x)e^{-x^2/2} = \frac{\sin(\xi)}{\sqrt{\pi}}$$

where $x = \xi/(2\sqrt{m})$ to prove

$$K_{2m}(x, y) = e^{-(x^2+y^2)} \sum_{j=0}^{2m-1} \pi_j(x)\pi_j(y) \tag{1}$$

converges to the sine kernel

$$2\sqrt{\pi} m \, \frac{\sin(x - y)}{x - y}.$$

2. (C) Do a numerical experiment to "see" the convergence in Problem 1. There are numerical issues on the diagonal and corners. Probably on the diagonal, Christoffel-Darboux needs to be replaced by a derivative approximation. See if you can make it better.

3. (C) Obtain the Airy Process limit by taking numerically

$$\frac{1}{\sqrt{2}\, n^{1/6}} K_n\left(\sqrt{2n} + \frac{x}{\sqrt{2}\, n^{1/6}}, \sqrt{2n} + \frac{y}{\sqrt{2}\, n^{1/6}}\right) \to \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x - y},$$

where $Ai(x)$ is the Airy function and K_n is defined in (1).