ASYMPTOTICS OF RANDOM LOZENGE TILINGS

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Abstract. This report is an exposition on the results that appear in the paper [VGGP]. We introduce the concept of a lozenge tiling, and state a few previous results in the field. We explain the relationship between random lozenge tilings and the asymptotics of certain symmetric polynomials, along with their ultimate connection to the eigenvalues of a finite dimensional GUE matrix.

1. Introduction and Previous Results on Lozenge Tilings

Let Ω be a domain drawn on the regular triangular lattice. A *lozenge tiling* is a tiling of the domain Ω by rhombi of three types. Each of these rhombi is a union of two adjacent triangles of the lattice (see Figure 1).

There is a correspondence between a lozenge tiling of a domain and a "stepped surface." A stepped surface is a piecewise linear surface in $\mathbb{R}^3$ that is formed from integer translates of the sides of a unit cube. By projecting this stepped surface along the $(1, 1, 1)$ direction, we recover a lozenge tiling (see Figure 2). If the stepped surface spans a boundary curve in $\mathbb{R}^3$ we recover a lozenge tiling on a domain Ω determined by the curve [Ke].

We may consider a probability measure on the space of these stepped surface as follows [OR1]. Let π be a stepped surface, and let $\mathrm{vol}(\pi)$ denote the volume enclosed by the surface — this can be defined up to an additive constant. We define

$$\mathbb{P}(\pi) = \frac{1}{Z(q)} q^{\mathrm{vol}(\pi)},$$

where $q > 0$ is a parameter and

$$Z(q) = \sum_{\pi} q^{\mathrm{vol}(\pi)},$$

is the normalization constant. We are interested in the limit that $q \to 1$: this is the case of uniformly distributed stepped surfaces (we approach constructing a uniform measure in this round-about way because if the region that we are constructing a stepped surface is unbounded, $Z(q)$ is finite only when $q < 1$). Also, we require that the limit $q \to 1$ is taken also with a rescaling of the mesh size of the lattice.

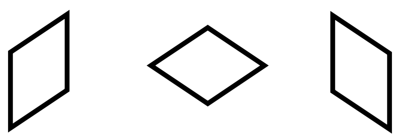

Figure 1. The 3 types of lozenges, the middle one is called "horizontal". This figure is taken from [VGGP].

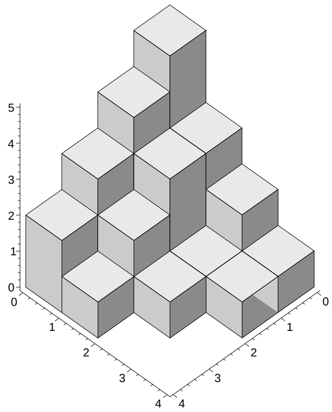

FIGURE 2. An example of a stepped surface, note the clear correspondence with a lozenge tiling. This figure is taken from [Ok].

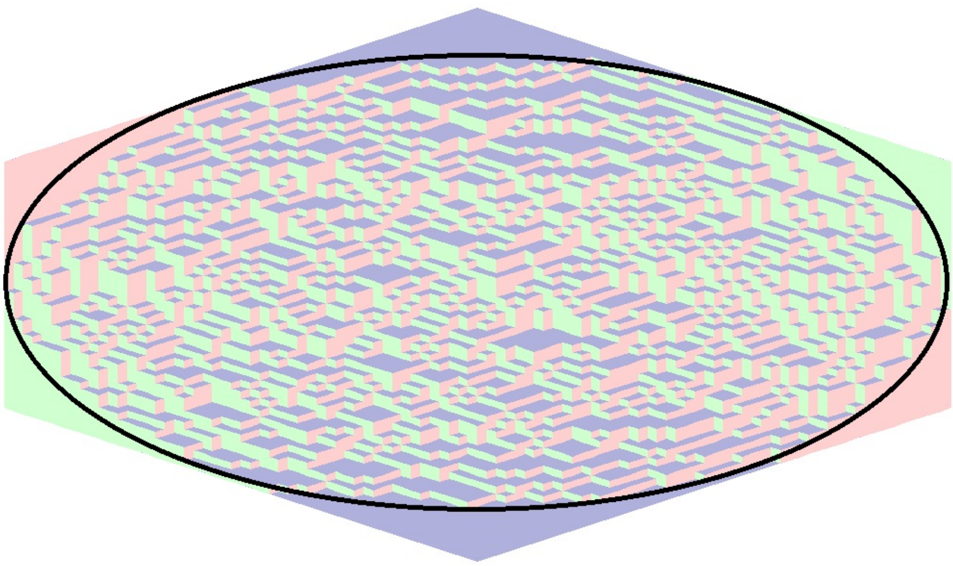

FIGURE 3. A sample from uniform distribution on tilings of $40 \times 50 \times 50$ hexagon and corresponding theoretical frozen boundary. The three types of lozenges are shown in three distinct colors. This figure is taken from [VGGP].

A famous result in the theory of random surfaces, is that in the limit as $q \to 1$, the stepped surface model above exhibits what is known as a

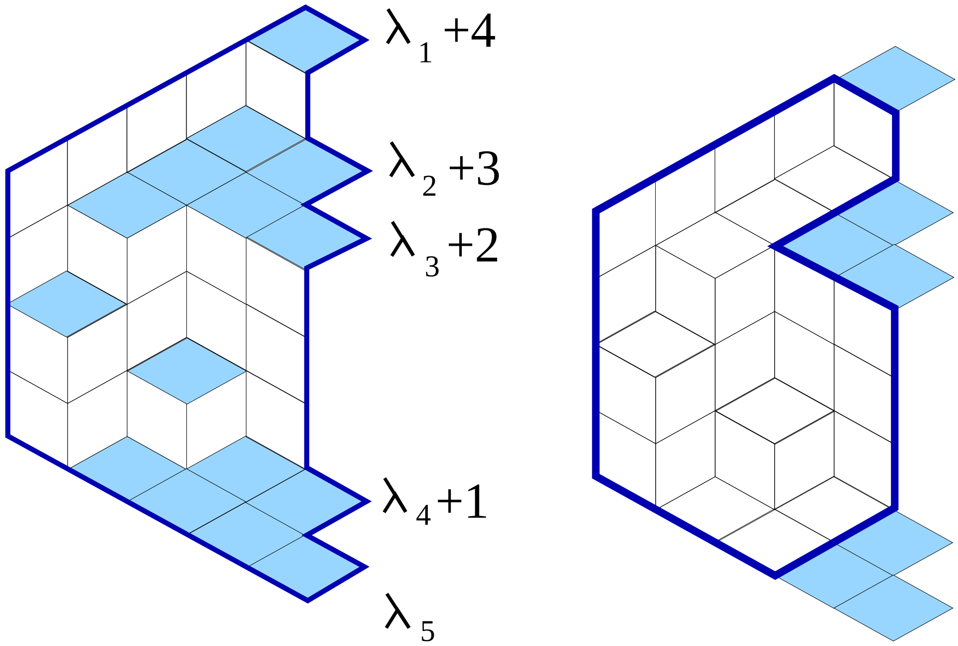

FIGURE 4. Lozenge tiling of the domain encoded by signature λ (left panel) and of corresponding polygonal domain (right panel highlighted in blue). This figure is taken from [VGGP].

"limit shape phenomenon." That is, as we take a random stepped surface on a lattice with mesh size shrinking, the random surface will grow closer to a fixed non-random surface. This should be viewed as a sort of "law of large numbers," for random surfaces. In particular, the stepped surfaces converge to a shape that has an ordered phase and a disordered phase with a boundary (the "limit shape") that separates them [Ke],[OR1]. In the case of a hexagonal domain Ω, the limit shape is an inscriped ellipse — for general polygonal domains the shape ends up being an inscribed algebraic curve (see Figure 3).

2. Combinatorics of Lozenge Tilings

We may encode a lozenge tiling of a domain in the following way (Figure 4 will be the reference for what follows). We compute the width of the domain by counting the total number N of horizontal lozenges that span from left to right of the domain. This can be done by imagining a vertical line passing through the center of each horizontal lozenge; we will count the total number of distinct vertical lines that is produced in this process as the width. Note that the left-most vertical line will intersect only one horizontal lozenge, while the second vertical line from the left intersects exactly two horizontal lozenges; it will follow that the k-th vertical line from the left intersects exactly k horizontal lozenges. In the figure, we see that the width $N = 5$.

The other piece of information needed to encode the tiling is N integers $\mu_1 > \mu_2 > \cdots > \mu_N$ which represent the positions of the horizontal lozenges sticking out of the right edge of the domain — these are simply a set of coordinates for the lozenges on the right hand side. For example in the figure $\mu_1 = 9$, $\mu_2 = 7$, $\mu_3 = 6$, $\mu_4 = 2$, and $\mu_5 = 1$, where we have started the coordinates from the bottom with increasing values upward (we can shift these values any way we like by a translation). If we rewrite $\mu_i = \lambda_i + N - i$, then $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N)$ is what is known as a *signature* of size N — which is a non-increasing sequence of integers of length N. Following our convention for μ_i in the figure, we have $\lambda_1 = 5$, $\lambda_2 = 4$, $\lambda_3 = 4$, $\lambda_4 = 1$, and $\lambda_5 = 1$. The domain is encoded by λ in the following way: we take the right-most horizontal lozenges on the diagram (whose positions, by construction, are given by λ) and remove all of them and those horizontal lozenges positioned near them due to boundary conditions — see the right hand side of 4 for an example.

We will denote domains encoded by a signature λ by Ω_λ. Further, we will let Υ_λ be a uniformly distributed random variable whose values are random tilings of the domain Ω_λ (there are finitely many such tilings so the distribution is well defined). Note that in our definition of Ω_λ, we removed all horizontal lozenges which were present deterministically due to the horizontal lozenges on the right with position specified by λ.

We now divert to discuss the combinatorics of signatures and partitions. A *partition* λ is a collection of non-negative numbers $\lambda_1 \geq \lambda_2 \geq \cdots$, such that $\sum_i \lambda_i < \infty$. A partition is also known as a *Young diagram* (we form a diagram by first creating a row of λ_1 squares side-by-side then directly below this row, starting from the left we draw λ_2 squares, etc…). These objects have significance in the representation theory of the symmetric group S_n (the symmetric group is the collection of all permutations of the integers $\{1, \ldots, n\}$).

Recall that a representation of a group G is a homomorphism $\rho : G \to GL(V)$ where V is some vector space. A representation ρ is irreducible if V is not trivial (i.e not $\{0\}$) and the only subspaces of V that are mapped to themselves under the action of every $\rho(g)$ are $\{0\}$ and V itself. In matrix notation this means that not all of the $\rho(g)$ can be decomposed into block matrices of the same form. Given a representation ρ, we define the character $\chi(g)$ of an element $g \in G$ to be $\mathrm{Tr}(\rho(g))$ — note that this does not depend on the basis chosen to write the matrix $\rho(g)$. Notationally speaking, a representation is also simply denoted by V, the vector space in question, and ρ can be omitted so that $gv = \rho(g)v$ for all $g \in G$ and $v \in V$.

The characters of a representation give information about the group. For example in the case of the symmetric group S_n, if we take what is known as the defining representation i.e. the representation in which we match each permutation to its corresponding permutation matrix, then the character of an element tells us about the number of fixed points of the permutation (the number of ones on the diagonal is the trace and it is the number of fixed points). Further, for S_n the set of irreducible representations corresponds to the number of partitions $\lambda_1 \geq \lambda_2 \geq \cdots$, such that $\sum_i \lambda_i = n$ [Sa].

We will denote the set of all signatures of size N as $\mathbb{GT}_N$. Note signatures $\lambda \in \mathbb{GT}_N$ are in correspondence with strict signatures $\mu_1 > \mu_2 > \cdots > \mu_N$ by the mapping $\mu_i = \lambda_i + N - i$ (this was the mapping we used previously).

Signatures are also useful in the study of the representations of other groups. In particular, the unitary group $U(N)$ has its irreducible representations V_λ parameterized by $\lambda \in \mathbb{GT}_N$. The character of an irreducible representation V_λ of $U(N)$ for a unitary matrix with eigenvalues $u_1, \dots, u_N$ are given by *Schur functions*:

$$s_\lambda(u_1, \dots, u_N) = \frac{\det \left[u_i^{\lambda_j + N - j} \right]_{i,j=1}^N}{\prod_{i<j}(u_i - u_j)}.$$

Notice that the denominator in the above function is a Vandermonde determinant, and that the function s_λ is symmetric in its variables (this means if we permute the variables we leave the function unaffected). Schur functions will have particular significance in what follows, so it will be necessary to introduce the following *normalized Schur function*:

$$S_\lambda(x_1, \dots, x_k; N, 1) = \frac{s_\lambda(x_1, \dots, x_k, \overbrace{1, \dots, 1}^{N-k})}{s_\lambda(\underbrace{1, \dots, 1}_{N})}.$$

To conclude this section we will define one more necessary operation. Given two Young diagrams μ and λ such that $\mu \subset \lambda$ in the sense that μ can be drawn inside of λ. The skew shape is the diagram λ/μ which is obtained by cutting out the squares of μ that can be contained in λ the squares that are left over from λ form the skew shape. Using the Schur function s_λ and s_μ we can define a *skew Schur functions* from the skew shape: $s_{\lambda/\mu}$. The way this is defined is by noting first that the Schur functions form a basis for the space of all symmetric functions. Then, we define an inner product that make s_λ, $\lambda \in \mathbb{GT}_N$ an orthonormal basis:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu},$$

we define $s_{\lambda/\mu}$ to be the unique symmetric function that satisfies

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle$$

for all partitions $\nu \in \mathbb{GT}_N$ [M].

3. Statement of the Results

The paper [VGGP] analyzes the local behavior of lozenge tilings near a *turning point* of the frozen boundary, which is the point where the frozen region is tagent to the boundary of the domain.

Using the notation of the previous section, we will let Ω_λ denote the domain encoded by $\lambda \in \mathbb{GT}_N$ and define Υ_λ to be a uniformly random lozenge tiling of the domain Ω_λ. We let $\nu_1 > \nu_2 > \cdots > \nu_k$ denote the coordinates of the horizontal lozenges at the kth vertical line from the left of our turning point. Using the mapping from the previous section we turn ν into a signature $\nu_i = \kappa_i + k - i$ and we obtain a random signature κ of size k, which we will represent with the variable Υ_λ^k.

There is a relationship between the Schur functions defined in the previous section and the distribution of Υ_λ^k which is given in the proposition below.

Proposition 3.1. *The distribution of Υ_λ^k is given by:*

$$\mathbb{P}\left(\Upsilon_\lambda^k = \eta\right) = \frac{s_\eta(1^k) s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

where $s_{\lambda/\eta}$ is the skew Schur polynomial.

Remark. The notation 1^k denote the k-tuple $(1, \dots, 1)$.

From this proposition it is clear that we should suspect that by knowing the asymptotics of Schur functions, we will be able to determine the asymptotics of Υ_λ^k in the limit that $N \to \infty$.

A $k \times k$ GUE random matrix X is a probability measure on the space of Hermitian matricies whose density is proportional to $\exp(-\mathrm{Tr}(X^2)/2)$. We will denote $\mathbb{GUE}_k$ as the distribution of k ordered eigenvalues of the GUE matrix. We have the following result on the asymptotics of Υ_λ^k.

Theorem 3.2 ([VGGP])**.** *Let $\lambda(N) \in \mathbb{GT}_N$, $N = 1, 2, \dots$ be a sequence of signatures. Suppose that there exist a non-constant piecewise-differentiable weakly decreasing function $f(t)$ such that*

$$\sum_{i=1}^{N} \left| \frac{\lambda_i(N)}{N} - f\left(\frac{i}{N}\right) \right| = o(\sqrt{N}),$$

as $N \to \infty$ and also $\sup_{i,N} |\lambda_i(N)/N| < \infty$. Then for every k as $N \to \infty$ we have

$$\frac{\Upsilon_{\lambda(N)}^k - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k$$

in the sense of weak convergence, where

$$E(f) = \int_0^1 f(t)\, dt, \quad S(f) = \int_0^1 f(t)^2\, dt - E(f)^2 + \int_0^1 f(t)(1-2t)\, dt.$$

Remark. For any non-constant weakly decreasing $f(t)$ we have $S(f) > 0$.

The paper [VGGP] obtained the following asymptotics of the Schur function as an application of the method of steepest descent.

Proposition 3.3 ([VGGP])**.** *Suppose that $f(t)$ is piecewise-differentiable, $\lambda(N) \in \mathbb{GT}_N$,*

$$\sup_{j=1,\dots,N} \left| \frac{\lambda_j(N)}{N} - f\left(\frac{j}{N}\right) \right| = O(1),$$

and

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left| \frac{\lambda_j(N)}{N} - f\left(\frac{j}{N}\right) \right| \to 0,$$

as $N \to \infty$. Then for any $h \in \mathbb{C}$,

$$S_{\lambda(N)}(e^{h/\sqrt{N}}; N, 1) = \exp\left(\sqrt{N}E(f)h + \frac{1}{2}S(f)h^2 + o(1)\right)$$

as $N \to \infty$, where $E(f)$ and $S(f)$ are as defined in Theorem 3.2. Moreover, we have that the remainder $o(1)$ is uniform over h belonging to compact subsets of $\mathbb{C}\backslash 0$.

Remark. This result is a combination of propositions 4.3 and 4.5 from [VGGP]. In proposition 4.3, only $h \in \mathbb{R}$ was dealt with; proposition 4.5 generalized the result to $h \in \mathbb{C}$.

In order to link the asymptotics of $\Upsilon^k_{\lambda(N)}$ with the Schur functions we need to define the multivariate normalized Bessel functions $B_k(x;y)$, $x = (x_1,\dots,x_k)$, $y = (y_1,\dots,y_k)$

$$B_k(x;y) = \frac{\det_{i,j=1,\dots,k}\Big(\exp(x_i y_j)\Big)}{\prod_{i<j}(x_i - x_j)\prod_{i<j}(y_i - y_j)}\prod_{i<j}(j-i).$$

The necessity of the normalized Bessel function is in the following result

Proposition 3.4 ([VGGP]). *Let* $\phi^N = (\phi_1^N \geq \phi_2^N \geq \cdots \geq \phi_k^N)$, $N = 1, 2, \dots$ *be a sequence of* k*-dimensional random variables. Suppose that there exists a random variable* ϕ^∞ *such that for every* $x = (x_1, \dots, x_k)$ *in a neighborhood of* $(0, \dots, 0)$ *we have*

$$\lim_{N\to\infty} \mathbb{E}B_k(x;\phi^N) = \mathbb{E}B_k(x;\phi^\infty).$$

Then $\phi^N \to \phi$ *in the sense of weak convergence of random variables.*

Remark. The above result amounts to a convergence of moment generating functions, as will be seen below.

The condition given in Proposition 3.4 is what will be used to obtain the weak convergence result of Theorem 3.2. We may see this by the following propositions

Proposition 3.5 ([VGGP]). *For* $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \in \mathbb{GT}_k$ *we have*

$$\frac{s_\lambda(e^{x_1},\dots,e^{x_k})}{s_\lambda(1^k)}\prod_{i<j}\frac{e^{x_i}-e^{x_j}}{x_i - x_j} = B_k(x_1,\dots,x_k;\lambda_1 + k - 1, \lambda_2 + k - 2\dots,\lambda_k)$$

Proposition 3.6 ([VGGP]). *We have*

$$\mathbb{E}B_k(x;\Upsilon^k_\lambda + \delta_k) = \frac{s_\lambda(e^{x_1},\dots,e^{x_k},1^{N-k})}{s_\lambda(1^N)}\prod_{1\leq i<j\leq k}\frac{e^{x_i}-e^{x_j}}{x_i - x_j},$$

where $\delta_k = (k-1, k-2, \dots, 0)$

Remark. The expression $\mathbb{E}B_k(x;\Upsilon^k_\lambda + \delta_k)$ is the moment generating function of Υ^k_λ. This becomes clearer for $k = 1$ where $\mathbb{E}B_k(x;\Upsilon^k_\lambda) = \mathbb{E}\exp(x\Upsilon^1_\lambda)$.

Thus from these results we need to compute

$$\mathbb{E}B_k(x;\mathbb{GUE}_k),$$

and compare with the limit of $\mathbb{E}B_k(x;\Upsilon^k_{\lambda(N)} + \delta_k)$ as $N \to \infty$. If we use the asymptotics for Schur functions in Proposition 3.3 along with the previous two propositions, and if we carry out a matrix integral for $\mathbb{E}B_k(x;\mathbb{GUE}_k)$ we obtain

$$\mathbb{E}B_k(x;\mathbb{GUE}_k) = \exp\left(\frac{1}{2}(h_1^2 + \cdots + h_k^2)\right) = \lim_{N\to\infty}\mathbb{E}B_k(x;\Upsilon^k_{\lambda(N)} + \delta_k),$$

which yields the required result: Theorem 3.2.

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