Plancherel Measure of Partitions and RMT

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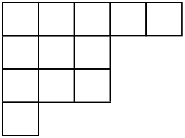
Young diagrams and SYT's

Young diagrams - partitions of n. Example $\lambda = (5, 3, 3, 1)$



Standard Young tableau, a way to fill the squares of a Young diagram: increasing by row and column.

1	2	3
4		

The RSK algorithm

The Robinson-Schensted correspondence is an algorithmic bijection between permutations and pairs of SYTs.

$$(3241) \longleftrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} .$$

Insert every element of the permutation into an initially empty tableau, keeping track of the growth of the first tableau in the second. The **shape** of the tableaux is the same.

$$n! = \sum_{\lambda} f_{\lambda}^{2}$$

Plancherel Measure on partitions

The pushforward of the uniform measure on permutations by the previous correspondence is the Plancherel measure on partitions. i.e.

$$\mathbb{P}(\text{shape is } \lambda) = \frac{f_\lambda^2}{n!}.$$

This measure is invariant by conjugation of the partition:

$$f_\lambda = \frac{n!}{\prod_{x \in T(\lambda)} h(x)} = f_{\lambda'}$$

Basic question: What is the distribution of λ? Vershik and Kerov proved that

$$\frac{\lambda_1}{\sqrt{n}} \to 2.$$

Okounkov's Results

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, normalize by

$$x_i = n^{1/3}\left(\frac{\lambda_i}{n^{1/2}} - 2\right)$$

For a GUE matrix H, with eigenvalues $E_1 \geq E_2 \geq \ldots$, normalize by

$$y_i = n^{2/3}\left(\frac{E_i}{n^{1/2}} - 2\right)$$

then in the limit as $n \to \infty$, and for any k,

$$(x_1, \ldots, x_k) \sim (y_1, \ldots, y_k).$$

Calculating Limiting distributions in RMT

Last ingredient: How to calculate limit distributions?

- Use a big matrix
- Solve a nonlinear differential equation.
- Fredholm determinants, Bornemann's Matlab package. Used to calculate limit level probabilities

$$E_2^{(n)}(k;J) = \mathbb{P}(\text{exactly } k \text{ eigenvalues of the } n \times n \text{ GUE lie in } J)$$

and

$$E_2(k;J) = \lim_{n\to\infty} E_2^{(n)}(k;\sqrt{2n} + 2^{-1/2}n^{-1/6}J).$$

Julia implementation

- Size of the structure: Use the expectation of λ_1 and symmetry.
- Julia and vectorization: Vectorization is bad, loops are good
- Binary Search on columns!
- Found a bug in rand
- The Julia project itself
- Pretty pictures!(Using Matlab)

- A. Okounkov, *Random Matrices and Random Permutations*, 2000
- A. Vershik and S. Kerov, *Asymptotics of the maximal and typical dimension of irreducible representations of symmetric group*, Func, Anal. Appl., **19**, 1985, no.1.
- F. Bornemann, *On the numerical evaluation of distributions in random matrix theory: A review*, 2000