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Course Project: Financial Applications of Random Matrix Theory

This paper will provide a brief overview of the uses of RMT in finance. The primary application of RMT in finance is improving estimates from empirical covariance matricies of various assets. This paper will begin with a brief overview of modern portfolio on which these applications rely. The paper will then outline the use of the Marcenko-Pastur law in analyzing empirical covariance matricies. A comparison of predictions from Marcenko-Pastur and the actual empirical data will be given and what conclusions can be drawn from the differences between the two. The paper will then briefly look at some other applications of RMT in finance.

The goal of portfolio optimization is to maximize expected returns while minimizing risk (defined as variance). This can be phrased as either fixing the risk and then maximizing the expected return or fixing the expected return and then minimizing the risk. We will work with the later formulation. Formally for a portfolio P of N assets we define the expected return R_P as $R_P = \sum_{i=1}^{N} p_i R_i$ where p_i is the amount invested in asset i (we normalize so that $\sum_{i=1}^{N} p_i = 1$) and R_i is the expected return of asset i. We define the risk as $\sigma_P^2 = \sum_{i,j=1}^{N} p_i C_{ij} p_j$ where C is the covariance matrix. For fixed R_P we can minimize σ_P with Lagrange multipliers. We get N+2 linear equations

for all i $2\sum_{j=1}^{N} p_j C_{ij} + \lambda_1 R_i + \lambda_2 = 0$

$\sum_{i=1}^{N} p_i R_i = R_P$

$\sum_{i=1}^{N} p_i = 1$

Unfortunately, getting C from empirical data is tricky. Our financial time series are generally not very long (the length is usually in the thousands) while the number of assets we want to consider are typically in the hundreds. This will introduce substantial noise in the covariance matrix. The general idea of this paper is to use the Marcenko-Pastur Law to estimate what the empirical covariance matrix if all assets were uncorrelated and then examine the discrepancy to find actual signal. We can define the empirical covariance matrix C by $C_{ij} = \frac{1}{T}\sum_{i=1}^{T} \delta x_i(t)\delta x_j(t)$ where the δx_is are normalized so that they have mean zero and variance one.

If we assume the assets are uncorrelated then the δx_is are independent, identically distributed random variables. Hence, C becomes a Wishart matrix or a Laguerre ensemble. We define $Q = T/N$ and if we hold Q constant while taking $N \to \infty, T \to \infty$ the Marcenko-Pastur law gives the distribution of eigenvalues of C: $\rho_C(\lambda) = \frac{Q}{2\pi\sigma^2}\frac{\sqrt{(\lambda_{max}-\lambda)(\lambda-\lambda_{min})}}{\lambda}$ for $\lambda \in [\lambda_{min}, \lambda_{max}]$

with $\lambda_{min}^{max} = \sigma^2(1 + 1/Q \pm \sqrt{1/Q})$

These results only strictly hold for $N \to \infty$ but they are approximately valid for much smaller N. We could explicitly look at the error term from using finite N but it does not add much to the discussion so we omit it. In a 1999 paper, Laloux et al compared the prediction based on uncorrelated asset returns to the performance of the S&P 500. They find that the largest eigenvalue based on the performance of the S&P 500 is about 25 times as large as the theoretical maximum which presumably indicates an actual signal. Indeed the corresponding eigenvector to this eigenvalue puts roughly equal weight on all stocks representing common market performance. However, the distribution of small eigenvalues is close to the predicted distribution and the vast majority of eigenvalues fall in $[\lambda_{min}, \lambda_{max}]$.

In a 2002 paper, Plerou et al analyze returns of a set of 1,000 stocks over three periods and find largely similar results. They also find that many of the other larger than predicted eigenvalues have corresponding eigenvectors that are largely made up of stocks from a particular industry. They also find that the distribution of spaces between eigenvalues is mostly in agreement with the predictions of random matrix theory. Furthermore, they find that the deviating eigenvectors are largely stable in time. In a 2004 paper, Utsugi et al find largely similar results for the Tokyo Stock Exchange although they do not find that large eigenvectors correspond to eigenvectors with components from particular industries as Plerou et al did.

Laloux et al also find that for eigenvectors corresponding to eigenvalues that fall in the predicted range their components follow a Gaussian distribution as predicted by random matrix theory. However, the components of eigenvectors corresponding to large eigenvalues do not follow this distribution. Laloux et al use this to "clean" the covariance matrix which they find reduces risk. However, they find that even the "cleaned" covariance matrix understates true risk.

In a 2004 paper, Pafka et al generalize the above techniques for exponentially weighted random matricies. The method is motivated both by the earlier work on applications of random matrix theory to financial covariance matricies discussed above and by other work on exponentially weighted averaging of returns to deal with heteroskedasticity in financial time series. There are no corresponding analytic results to those found above but numerical results can be readily found. Pafka et al find that the combination of random matrix theory filtering of data and exponential weighting of returns produces better estimates than either method alone.

In conclusion, results from random matrix theory are of great interest but theoretically and practically in understanding financial time series. Although the principal application of random matrix theory in finance remains reducing noise in empirical covariance matricies there are other potential applications as well. A 2009 paper by Eom et al uses random matrix theory to help determine topological properties of financial networks based on minimal spanning trees. A 2009 paper by Bai et al uses random matrix theory and bootstrapping to improve portfolio optimization.

References

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