18.338 Final Report: The distribution of zeros of the derivative of a random polynomial

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I. Introduction

There has been significant interest on the zeros of the derivative of polynomials, known as critical points, when the zeros of the polynomial were known. According to the Gauss-Lucas theorem [2], the critical points of a polynomial f lie in the convex hull of the zeros of f. In this project, I verified the experiments done in a probabilistic study of the critical points of polynomials done in [1], using the Matlab programming environment. Following [1], I present Jensen's theorem and Marden's theorem in Section II. While these theorems address the location of non-real critical points of a polynomial, there has not been any probabilistic study of critical points; that is a study on the critical points of a polynomial, whose roots follow some distribution. The authors address the question, when the zeros of f_n' are stochastically similar to the zeros of f_n . They conjecture that the critical points converge in distribution to the distribution of the roots of the polynomial, as the number of roots goes to infinity. They support this intuition with two experiments, one where the roots of the polynomial are distributed in the unit disk, and one where the roots are distributed in the unit circle. The experiments are described in Section III. While results show that the distribution of the critical points approximates the one of the roots, an exception to this case is shown in Section IV. Finally, the Matlab code for the simulations is presented in the Appendix.

II. Gauss-Lucas Theorem Extensions

A. Jensen's theorem: If $p(z)$ has real coefficients, then the non-real critical points of p lie in the union of the "Jensen Disks", which are disks one of whose diameters is the segment joining a pair of conjugate (non-real) roots of p.

I implemented an example case in Matlab, and I present the result in Figure 1. I generated randomly 10 real numbers which are the coefficients of a polynomial. The blue stars in the figure are the roots of the polynomial, and the red stars are the critical points.

B. Marden's theorem: Suppose the zeros z_1, z_2, and z_3 of a third degree polynomial $p(z)$ are non-collinear. There is a unique ellipse inscribed in the triangle with vertices z_1, z_2, and z_3 and tangent to the sides at their midpoints: the Steiner inellipse. The foci of that ellipse are the zeros of the derivative $p'(z)$.

I implemented an example case in Matlab, where I arbitrarily picked three numbers as the coefficients of the polynomial, so that its roots are non-collinear. I then derived the lines connecting the three roots, so that they form a triangle. Using an open-source software [3], I derived the ellipse inscribed by the triangle and tangent to the sides at the triangle vertices' midpoints. The zeros of the derivative are inside the inscribed ellipse, and approximate the foci of the ellipse with some error. The true foci should be closer to the center of the ellipse, and the error in this approximation is left for future work.

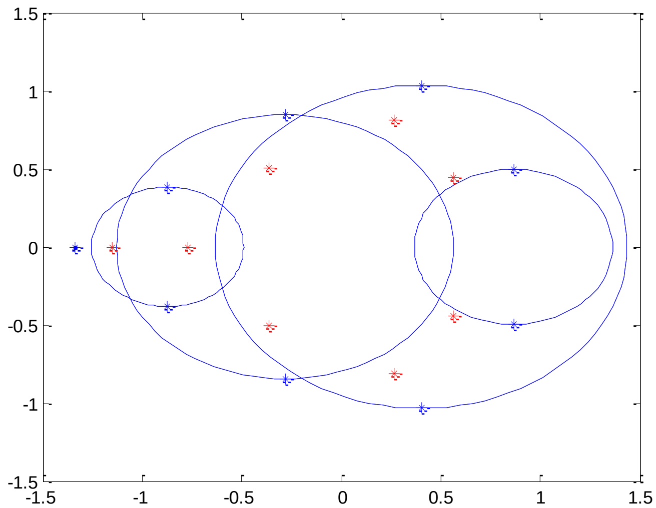

Figure 1: Jensen's theorem example

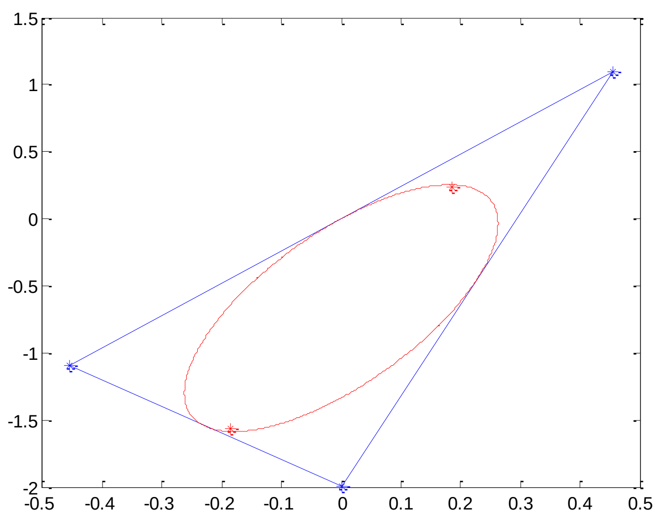

Figure 2: Marden's theorem example

III. Critical points of polynomials with random roots.

A. Problem Definition

Although there has been a study of the critical points, there has not been yet a probabilistic study of critical points. Previous work on random polynomials has focused on polynomials with coefficients as random variables, but what we focus on the case when the roots of the polynomial are drawn from some probability distribution. In [1], the authors define the problem as follows.

Let μ be a probability measure on the complex numbers. Let X_n be random variables on a probability space (Ω, F, P) that are IID with common distribution μ. Let $f_n(z) := \prod_{j=1}^{n}(z - X_j)$ be the random polynomial whose roots are $X_1, \ldots, X_n$. The question addressed in [1] is, *when are the zeros of f_n' stochastically similar to the zeros of f_n ?* The intuition behind this question is that if μ concentrates on real numbers, f_n has all real zeros and the zeros of f_n' interlace the zeros of f_n . Therefore, the empirical distribution of the zeros of f_n' converges to μ as $n \rightarrow \infty$.

The authors conjecture that for any μ, as $n \rightarrow \infty$, $Z(f')$ converges weakly to μ. To support this claim, they did the following two experiments.

B. Experiment 1: Roots of f uniformly distributed on the unit disk. We want to test, following [1], whether the critical points of f will approximate a uniform distribution on the unit disk, as well. We sampled 100 complex roots of the polynomial. We then calculated the critical points and plotted them in Figure 3. We see that there is some structure, and the viewer may or may not be convinced that the roots are uniformly sampled. We then increased the number of polynomial degree to 300, and plot the critical points in Figure 4. We see that the result is wrong, and we suspect that this is because of numerical issues in the way that Matlab calculated the roots of a polynomial of very large degree. Instead, we used the MuPad symbolic language [4], that is an add-on to the Mathworks Symbolic Math Toolbox. We plot the critical points in Figure 5, and see that there are no errors due to numerical precision issues.

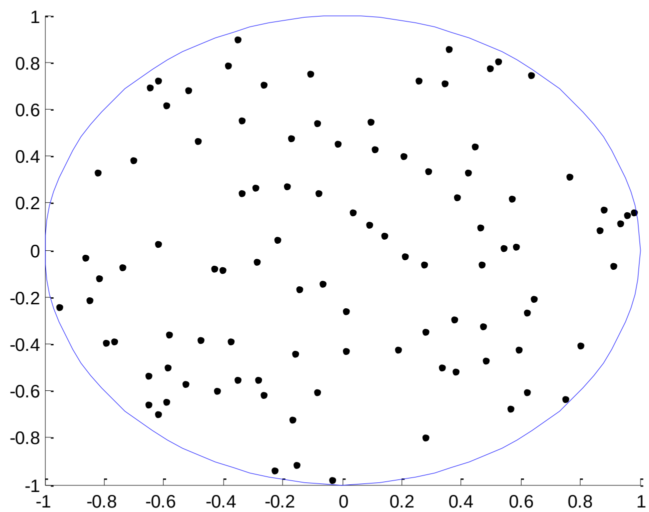

Figure 3: Critical points of polynomial of degree 100 whose roots are uniformly sampled inside unit disk.

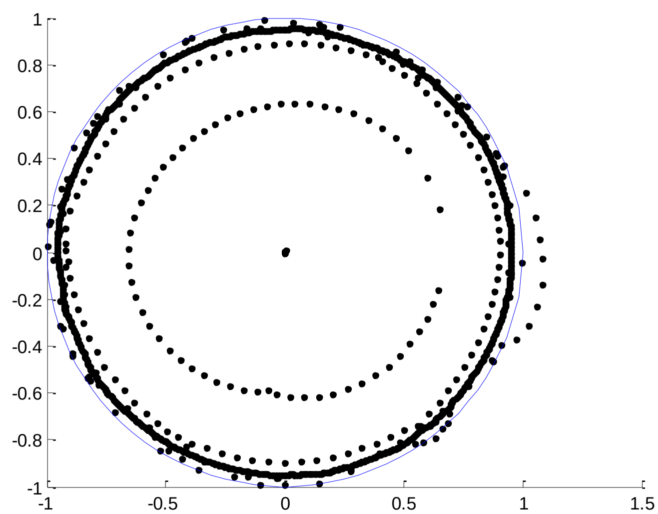

Figure 4: Critical points of polynomial of degree 1000, whose roots are uniformly sampled inside unit disk, plotted by Matlab, where numerical errors affect the result.

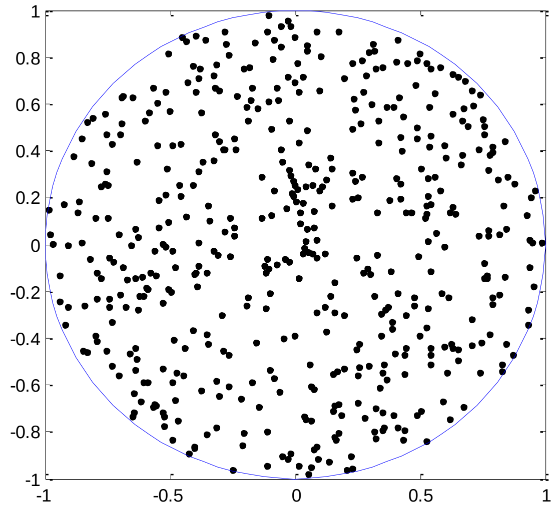

Figure 5: Critical points of polynomial of degree 500, whose roots are uniformly sampled inside unit disk, plotted using MuPad.

C. Experiment 2: Roots of *f* uniformly distributed on the unit circle. We want to test, following [1], whether the critical points of *f* will approximate a uniform distribution on the unit circle, as well. We sampled 100 complex roots of the polynomial. We then calculated the critical points and plotted them in Figure 6. We verify the result of [1], that shows quick convergence, with the presence of a few outliers.

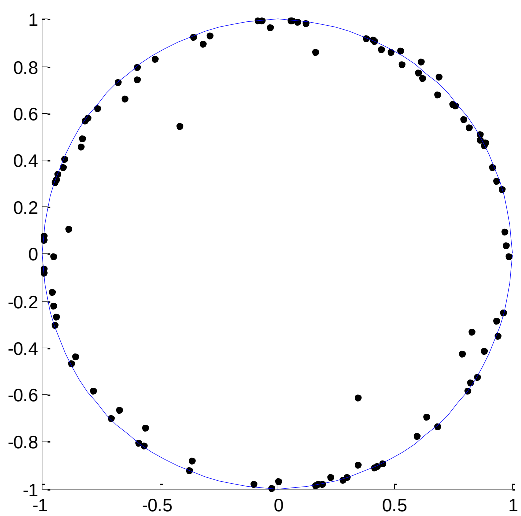

Figure 6: Critical points of polynomial of degree 100, whose roots are uniformly sampled inside unit circle, plotted using Matlab.

IV. Exceptional Cases

The authors finally provide a counterexample, that shows that one would need to rule out at least some exceptional sets of low probability. The authors give the example of $f(z) = z^n - 1$, whose roots converge to the uniform distribution on the unit circle, as $n \rightarrow \infty$. The roots for n = 10 are plotted in Figure 7.

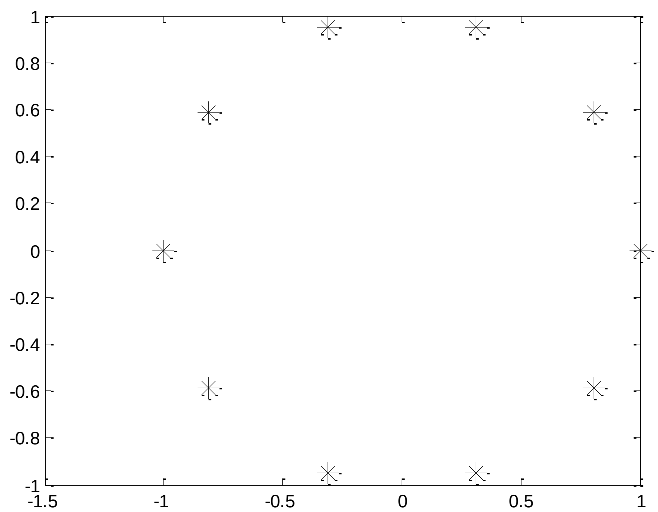

Figure 7: Roots of polynomial $\boldsymbol{f(z) = z^{10} - 1}$

The roots of the derivative of the polynomial are all concentrated at the origin. If one moves one of the roots of $f(n)$ along the unit circle, until it meets the next root, a distance of order $\frac{1}{n}$, the one root of f_n' zooms from the origin out to the unit circle. We verified this result, by moving one of the roots annotated in red upwards along the segment that connects it to the other root annotated in red (Figure 8). For different values of perturbation, we plot the critical points in Figure 9.

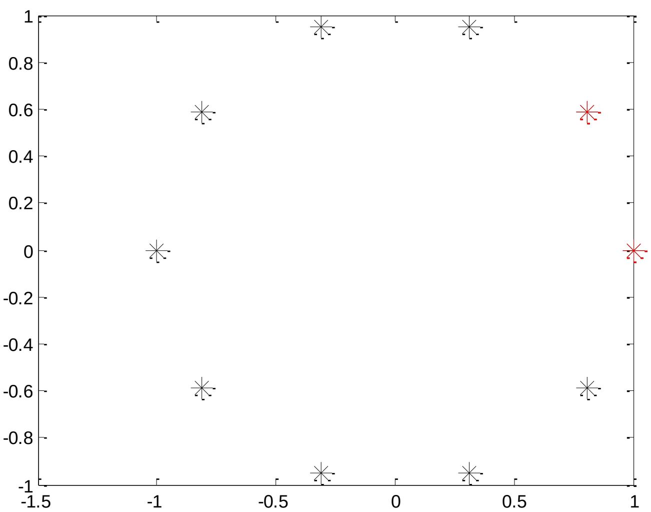

Figure 8: The lower root annotated in red moves towards the other root in red, and for different motions of the root, we plot the critical points of the polynomial f_n' in Figure 9.

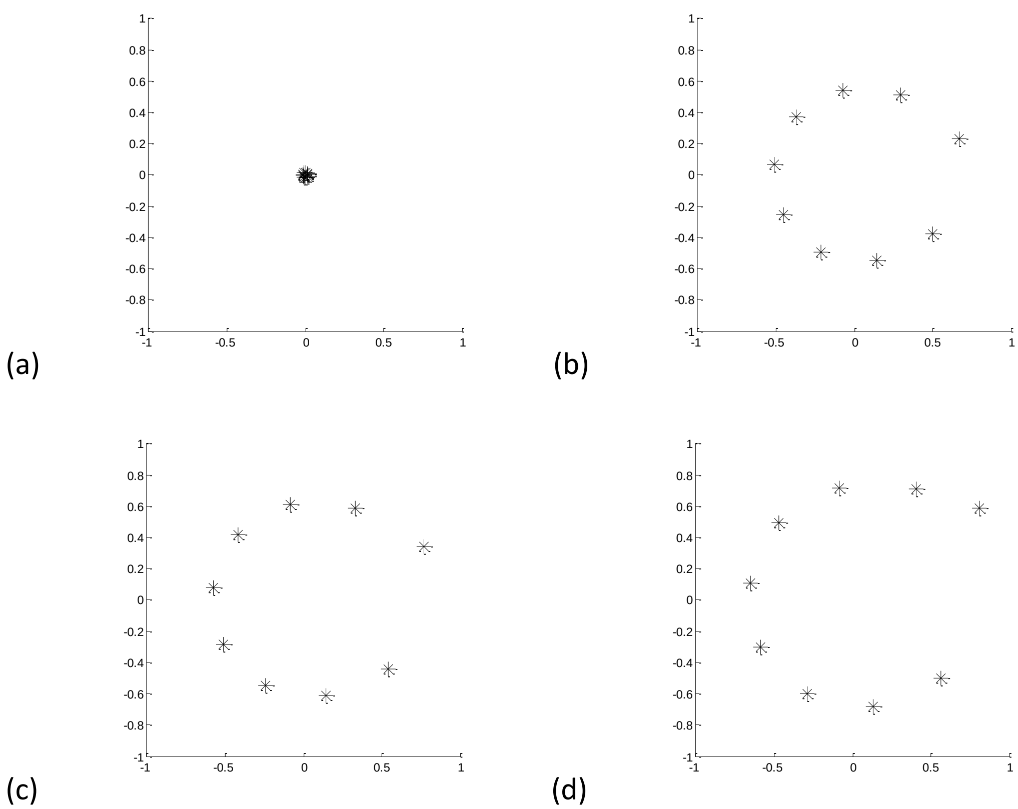

Figure 9: a) Critical points of f_n' for $(z) = z^{10} - 1$. b) Critical points after one root of f_n moves towards another root 1/10 of a distance, c) 3/10 of a distance, and d) the root of f_n becomes equal to another root.

Appendix

A. Jensen's Theorem Matlab Code

JensenTheorem.m

```
p = rand(1,10);
rootP = roots(p);
figure();
plot(rootP,'*')
rows = size(rootP,1);

hold on;
for t = 1:rows
    x(t) = real(rootP(t));
    r(t) = abs(imag(rootP(t)));
    circle(x(t),0,r(t));
end

dp = polyder(p);
rootDP = roots(dp);
rows = size(rootDP, 1);
hold on;
clear x
plot(rootDP,'*', 'color', 'r')
```

----

circle.m

```
function h = circle(x,y,r)

hold on
th = 0:pi/50:2*pi;
xunit = r * cos(th) + x;
yunit = r * sin(th) + y;
h = plot(xunit, yunit);
hold off
```

B. Marden's Theorem Matlab Code

MardensTheorem.m

```
%This code requires the MVE free software from:
http://www.caam.rice.edu/~zhang/mve/index.html
p = [1 +2*i 1 - 1*i 2+2*i]
rootP = roots(p);
x = real(rootP);
y = imag(rootP);
figure()
plot(rootP,'*')
hold on
line(x,y)
line([x(3) x(1)],[y(3) y(1)])


%writing triangle lines in from Ax = b
sign1 = -1;
lambda1 = (y(2)-y(1))/(x(2)-x(1));
```

```
A(1,:) = sign1*[-lambda1, 1];
b(1) = sign1*(y(1)-lambda1*x(1));


sign2 = -1;
lambda2= (y(3)-y(2))/(x(3)-x(2));
A(2,:) = sign2*[-lambda2, 1];
b(2) = sign2*(y(2)-lambda2*x(2));

sign3 = 1;
lambda3 = (y(1)-y(3))/(x(1)-x(3));
A(3,:) = sign3*[-lambda3, 1];
b(3) = sign3*(y(3)-lambda3*x(3));

x0 = [0;-1];

[x,E] = mve_run(A,b',x0);

fprintf('  drawing ..........\n')
draw_ellipse(A,b,x0,x,E);

DP = polyder(p);
rootsDP = roots(DP);

plot(rootsDP, '*', 'color', 'r')
```

C. Experiment 1 Matlab Code

Exp1.m

```
x1=0;
y1=0;
rc = 1;
for t = 1:N
    [a(t) b(t)] = cirrdnPJ(x1,y1,rc);
end
figure;

A = diag([a + b*i]);

p = poly(A);
dp = polyder(p);

rootsDp = roots(dp);

MS = 'markersize'; FS = 'fontsize'; ms = 12; fs = 12;
plot(rootsDp,'.k',MS,ms)
circle(0,0,1);
```

----

cirrdnPJ.m

```
function [x y]=cirrdnPJ(x1,y1,rc)
a=2*pi*rand;
r=sqrt(rand);
x=(rc*r)*cos(a)+x1
y=(rc*r)*sin(a)+y1
end
```

D. Experiment 2 Matlab Code

Exp2.m

```
N = 100;

x1=0;
y1=0;
rc = 1;
for t = 1:N
    [a(t) b(t)] = circRG(x1,y1,rc);
end
figure;

A = diag(a + b*i);
p = poly(A);
dp = polyder(p);

hold on;
rootsDp = roots(dp);

MS = 'markersize'; FS = 'fontsize'; ms = 12; fs = 12;
plot(rootsDp,'.k',MS,ms)
axis([-1 1 -1 1]), axis square
circle(0,0,1);
```

----

circRG.m

```
function [x y]=circRG(x1,y1,rc)
a=2*pi*rand;
x=(rc)*cos(a)+x1
y=(rc)*sin(a)+y1
end
```

References

[1] Pemantle, Robin, and Igor Rivin. "The distribution of zeros of the derivative of a random polynomial." *arXiv preprint arXiv:1109.5975*(2011).

[2] M. Marden. Geometry of Polynomials, volume 3 of Mathematical Surveysand Monographs. AMS, 1949.

[3] Maximum Volume Ellipsoid free software, http://www.caam.rice.edu/~zhang/mve/index.html

[4] Mupad http://www.mathworks.com/discovery/mupad.html