Random Curve and Surface

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Abstract

In this project, we use the determinantal process tools to study random partition, which naturally comes from Ulam problem, and explain why random matrices processes naturally arise. In particular, we give a proof of Baik-Deift-Johansson theorem. Then we generalize the mechanism to study random surface, using Schur process. We will see the frozen boundary phenomenon, limit shape, and different process related to random matrices near different points.

1 Ulam Problem, Planchele Measure and Poissonization

Let σ be a uniformly distributed permutation of the set $\{1,\dots,n\}$, and let $\ell_n(\sigma)$ be the length of the longest increasing subsequence of σ.The problem of understanding the limit behavior of ℓ_n has a long history and goes back to the book of Ulam of 1961 [Ulam-61].Ulam conjectured that $\mathbb{E}\ell_n \approx c\sqrt{n}$ but was not able to identify the constant; he also conjectured Gaussian fluctuations. In 1974 Hammersley [Hammersley-72] proved, via a sub–additivity argument, that there exists a constant such that $\ell_n \approx c\sqrt{n}$ and this constant was identified in 1977 by Kerov and Vershik [Vershik-Kerov-77]. However, the fluctuation was not understood until the ground breaking work of Baik, Deift and Johansson, the completely answered Ulam's question. They show that

Theorem 1.1 ([Baik-Deift-Johansson-99]). *Let σ be a uniformly distributed permutation of the set $\{1,\dots,n\}$, and let $\ell_n(\sigma)$ be the length of the longest increasing subsequence of σ. Then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell_n - 2\sqrt{n}}{n^{1/6}} \le s\right) = F_2(s),$$

where $F_2(s)$ is GUE Tracy-Widom law.

By RSK correspondence, uniform measure on permutation group become Planchele measure on partition, which is $\mathbb{P}(\lambda) = \frac{\mathrm{dim}\lambda^2}{n!}$, where λ is a partition of n and $\mathrm{dim}\lambda$ is the dimension of the irreducible representation of S_n indexed by λ. $\mathrm{dim}\lambda$ is also the number of standard Young tableaux of shape λ. The length of the longest subsequence correspond to the length of the first row of the corresponding partition.

If we draw the partition in Russian style, and rescale by $\frac{1}{\sqrt{n}}$, we get a random curve, illustrated in figure 3. So we regarded Plancherel measure as a random curve. As we will see, this curve have a frozen boundary $\{2, -2\}$, and between $(-2, 2)$, there is a nontrivial limit shape, and at any point in the bulk $(-2, 2)$, in a window of scale $\frac{1}{\sqrt{n}}$, we will see a determinantal process with discrete sine kernel. At the edge , we will see Airy process, which is the as same as edge process of GUE. This random partition-random matrices correspondence is called Baik-Deift-Johansson(BDJ) conjecture, which is proved by [Okounkov00], [Borodin-Okounkov-Olshanski-00] and [Johansson-01a].

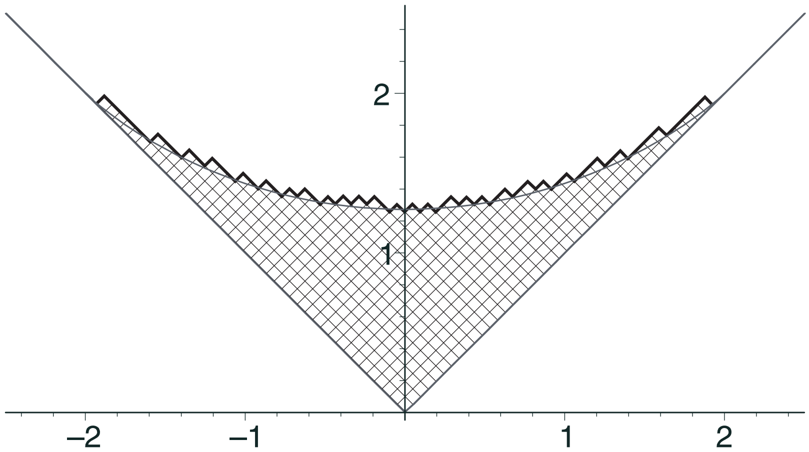

Figure 1:

The original proof of BDJ conjecture in [Okounkov00] is moment method plus very fine combinatorial argument. But that argument fails to give information in the bulk. Here we discuss this problem in a more general framework which has rich algebraic structure. This general framework help us fully understand this random curve and further more, its 2 dimensional generalization, random surface.

First of all, we Poissonize our random partition. We first choose a n according to Poisson(θ^2), then sample a random partition according to

Plancherel measure of partitions of n. Then we get the Poissonized Plancherel measure

$$\mathbb{P}(\lambda(\theta) = \mu) = e^{-\theta^2} \left(\frac{\theta^{|\mu|} \dim(\mu)}{|\mu|!} \right)^2, \quad \mu \in \mathbb{Y}. \tag{1.1}$$

Poissonized Planchele measure is a measure defined on all the partitions, or in another word, on Young diagram . Since Poisson(θ^2) has expectation θ^2 and standard deviation θ, as θ tends to ∞, the measure will concentrate near $n = \theta^2$. So heuristically we can reduce the asymptotic of Planchele measure to asymptotic of Poissonized Planchele measure. For the rigour treatment of this equivalence, we refer to [Borodin-Okounkov-Olshanski-00].

2 Schur measure and its determinantal structure

To study Plancherel measure, we are going to develop a general theory of an object called Schur measure. Schur measure is defined and studied in [Okounkov-01]. To define Schur measure, we first introduce the concept specialization of symmetric functions.

Let $\Lambda_N = \mathbb{C}[x_1, \ldots, x_N]^{S_N}$ be the space of polynomials in $x_1, \ldots, x_N$ which are symmetric with respect to permutations of the x_j. Λ_N has a natural grading by the *total degree* of a polynomial.

Let $\pi_{N+1} \colon \mathbb{C}[x_1, \ldots, x_{N+1}] \to \mathbb{C}[x_1, \ldots, x_N]$ be the map defined by setting $x_{N+1} = 0$. It preserves the ring of symmetric polynomials and gradings. Thus we obtain a tower of graded algebras

$$\mathbb{C} \xleftarrow{\pi_1} \Lambda_1 \xleftarrow{\pi_2} \Lambda_2 \xleftarrow{\pi_3} \ldots .$$

We define Λ as the *projective limit* of the above tower

$$\Lambda = \varprojlim_N \Lambda_N = \{(f_1, f_2, f_3, \ldots) \mid f_j \in \Lambda_j,\ \pi_j f_j = f_{j-1},\ \deg(f_j) \text{ are bounded }\}.$$

An equivalent definition Λ is as follows: Elements of Λ are formal power series $f(x_1, x_2, \ldots)$ in infinitely many indeterminates $x_1, x_2, \ldots$ of bounded degree that are invariant under the permutations of the x_i's. In particular,

$$x_1 + x_2 + x_3 + \ldots$$

is an element of Λ, while

$$(1 + x_1)(1 + x_2)(1 + x_3) \cdots$$

is not, because here the degrees are unbounded.

Elementary symmetric functions e_k, $k = 1, 2, \dots$ are defined by

$$e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} \cdots x_{i_k}.$$

Complete homogeneous functions h_k, $k = 1, 2, \dots$ are defined by

$$h_k = \sum_{i_1 \le i_2 \le \cdots \le i_k} x_{i_1} \cdots x_{i_k}.$$

Power sums p_k, $k = 1, 2, \dots$ are defined by

$$p_k = \sum_i x_i^k.$$

Definition 2.1. *The* Schur polynomial $s_\lambda(x_1, \dots, x_N)$ *is a symmetric polynomial in* N *variables parameterized by Young diagram* λ *with* $\ell(\lambda) \le N$ *and given by*

$$s_\lambda(x_1, \dots, x_N) = \frac{\det \left[x_i^{\lambda_j + N - j} \right]_{i,j=1}^N}{\prod_{i<j}(x_i - x_j)}. \tag{2.1}$$

One proves that when $\ell(\lambda) \le N$

$$\pi_{N+1} s_\lambda(x_1, \dots, x_N, x_{N+1}) = s_\lambda(x_1, \dots, x_N, 0) = s_\lambda(x_1, \dots, x_N).$$

In addition,

$$\pi_{\ell(\lambda)} s_\lambda(x_1, \dots, x_{\ell(\lambda)}) = 0.$$

Therefore, the sequence of symmetric polynomials $s_\lambda(x_1, \dots, x_N)$ with fixed λ and varying number of variables $N \ge \ell(\lambda)$, complemented by zeros for $N < \ell(\lambda)$, defines an element of Λ that one calls the *Schur symmetric function* s_λ. By definition $s_\varnothing(x) \equiv 1$.

Definition 2.2. *Let* λ *be any Young diagram. Expand* $s_\lambda(x, y)$ *as a linear combination of Schur symmetric functions in variables* y_i*; the coefficients of this expansion are called* skew Schur functions *and denoted* $s_{\lambda/\mu}$*:*

$$s_\lambda(x, y) = \sum_\mu s_{\lambda/\mu}(x) s_\mu(y).$$

In particular, $s_{\lambda/\mu}(x)$ *is a symmetric function in variables* x_i*.*

We assume that the readers are familiar basic symmetric function, including Schur symmetric function and skew Schur symmetric function and various identities such like Chaucy identity. For those who are not, a standard reference is the first chapter of [Macdonald-95].

Definition 2.3. *Any algebra homomorphism* $\rho : \Lambda \to \mathbb{C}, \quad f \mapsto f(\rho)$, *is called a* specialization. *In other words,* ρ *should satisfy the following properties:*

$$(f+g)(\rho) = f(\rho) + g(\rho), \quad (fg)(\rho) = f(\rho)g(\rho), \quad (\theta f)(\rho) = \theta f(\rho), \quad \theta \in \mathbb{C}.$$

Take any sequence of complex numbers $u_1, u_2, \dots$ satisfying $\sum |u_i| < \infty$. Then the *substitution* map $\Lambda \to \mathbb{C}$, $x_i \mapsto u_i$ is a specialization. More generally, any specialization is uniquely determined by its values on any set of generators of Λ. Furthermore, if the generators are algebraically independent, then these values can be any numbers. What this means is that defining ρ is equivalent to specifying the set of numbers $p_1(\rho)$, $p_2(\rho)$,..., *or* the set of numbers $e_1(\rho)$, $e_2(\rho)$,..., *or* the set of numbers $h_1(\rho)$, $h_2(\rho)$,.... In particular, if ρ is the substitution of complex numbers u_i, then

$$p_k \mapsto p_k(\rho) = \sum_i (u_i)^k. \tag{2.2}$$

Note that the condition $\sum_i |u_i| < \infty$ implies that the series in (2.2) converges for any $k \geq 1$.

We call a specialization ρ *Schur–positive* if for every Young diagram λ we have

$$s_\lambda(\rho) \geq 0.$$

There is an explicit classification for Schur–positive specializations.

Theorem 2.4. *The Schur–positive specializations are parameterized by pairs of sequences of non-negative reals* $\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq 0)$ *and* $\beta = (\beta_1 \geq \beta_2 \geq \dots \geq 0)$ *satisfying* $\sum_i(\alpha_i + \beta_i) < \infty$ *and an additional parameter* $\gamma \geq 0$. *The specialization with parameters* $(\alpha; \beta; \gamma)$ *can be described by its values on power sums*

$$p_1 \mapsto p_1(\alpha; \beta; \gamma) = \gamma + \sum_i (\alpha_i + \beta_i),$$

$$p_k \mapsto p_k(\alpha; \beta; \gamma) = \sum_i (\alpha_i^k + (-1)^{k-1}\beta_i^k), \quad k \geq 2$$

or, equivalently, via generating functions

$$\sum_{k=0}^{\infty} h_k(\alpha; \beta; \gamma) z^k = e^{\gamma z} \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z}.$$

Theorem 2.4 is a deep theorem related to representation theory of infinite permutation group $S(\infty)$. The first proofs were obtained (independently) by Thoma [Thoma-64] and Edrei [Edrei-53], a proof by a different method can be found in [Vershik-Kerov-81], [Kerov-03], [Kerov-Okounkov-Olshanski-98], and yet another proof is given in [Okounkov-94].

Take any two Schur–positive specializations ρ_1, ρ_2 of the algebra of symmetric functions Λ (those were classified in Theorem 2.4). The following definition first appeared in [Okounkov-01].

Definition 2.5. The Schur measure $\mathbb{S}_{\rho_1;\rho_2}$ *is a probability measure on the set of all Young diagrams defined through*

$$\mathbb{P}_{\rho_1,\rho_2}(\lambda) = \frac{s_\lambda(\rho_1)s_\lambda(\rho_2)}{H(\rho_1;\rho_2)},$$

where the normalizing constant $H(\rho_1;\rho_2)$ *is given by*

$$H(\rho_1;\rho_2) = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(\rho_1)p_k(\rho_2)}{k}\right).$$

Remark. The above definition makes sense only if ρ_1, ρ_2 are such that

$$\sum_\lambda s_\lambda(\rho_1)s_\lambda(\rho_2) < \infty, \tag{2.3}$$

and in the latter case this sum equals $H(\rho_1;\rho_2)$. The convergence is guaranteed, for instance, if $|p_k(\rho_1)| < Cr^k$ and $|p_k(\rho_2)| < Cr^k$ with some constants $C > 0$ and $0 < r < 1$. In what follows we assume that this (or a similar) condition is always satisfied.

Proposition 2.6. *Let* ρ_θ *be the (Schur–positive) specialization with single non-zero parameter* $\gamma = \theta$*, i.e.*

$$p_1(\rho_\theta) = \theta, \quad p_k(\rho_\theta) = 0, \quad k > 1.$$

Then $\mathbb{P}_{\rho_\theta,\rho_\theta}$ *is the Poissonized Plancherel measure* (1.1).

Schur measure is a determinantal point process. Given a Young diagram λ, we associate to it a point configuration $X(\lambda) = \{\lambda_i - i + 1/2\} \subset \mathbb{Z} + 1/2$. Note that $X(\lambda)$ is semi–infinite, i.e. there are finitely many points to the right of the origin, but almost all points to the left of the origin belong to $X(\lambda)$.

Theorem 2.7 ([Okounkov-01]). *Suppose that the $\lambda \in \mathbb{Y}$ is distributed according to the Schur measure $\mathbb{S}_{\rho_1;\rho_2}$. Then $X(\lambda)$ is a determinantal point process on $\mathbb{Z}+1/2$ with correlation kernel $K(i,j)$ defined by the generating series*

$$\sum_{i,j\in\mathbb{Z}+\frac{1}{2}} K(i,j)v^i w^{-j} = \frac{H(\rho_1;v)H(\rho_2;w^{-1})}{H(\rho_2;v^{-1})H(\rho_1;w)} \sum_{k=\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots} \left(\frac{w}{v}\right)^k, \tag{2.4}$$

where

$$H(\rho;z) = \sum_{k=0}^{\infty} h_k(\rho)z^k = \exp\left(\sum_{k=1}^{\infty} p_k(\rho)\frac{z^k}{k}\right).$$

Sketch of proof. The two sides of 2.4 can been seen as the specialization ρ_1,ρ_2 the two formal symmetric function. So the only thing we need to show is the formal identity of symmetric function. It is a matter of comparing the coefficients. So we just need to deal will the case of finite variable specialization $\rho_1 = (x_1,\cdots,x_N), \rho_2 = (y_1,\cdots,y_N)$. But by definition of Schur polynomial, it involves the determinant of polynomials. Actually it falls in the class of bi-orthogonal ensembles which was introduced in [Bor98], a generalization of GUE, which we can use Chaucy-Bonnet formula to compute the correlation function. For details of this approach, see [BG12]. □

Apply 2.7 to Poissonized Plancherel measure, we get

Corollary 2.8 ([Borodin-Okounkov-Olshanski-00],[Johansson-01a]). *Suppose that λ is a random Young diagram distributed by the Poissonized Plancherel measure. Then the points of $X(\lambda)$ form a determinantal point process on $\mathbb{Z}+\frac{1}{2}$ with correlation kernel*

$$K_\theta(i,j) = \frac{1}{(2\pi\mathbf{i})^2}\oint\oint \exp\left(\theta(v-v^{-1}-w+w^{-1})\right)\frac{\sqrt{vw}}{v-w}\frac{dvdw}{v^{i+1}w^{-j+1}},$$

with integration over positively oriented simple contours enclosing zero and such that $|w|<|v|$.

Now the local asymptotic information of the random line giving by Plancherel measure is governed by the asymptotic behavior of $K_\theta(i,j)$.

The argument below is due to Okounkov [Okounkov-03], but the results were obtained earlier in [Borodin-Okounkov-Olshanski-00], [Johansson-01a] by different tools. Let us start from the case $i=j$. Then $K(i,i)$ is the *density* of particles of our point process, or the average local slope of the (rotated) Young diagram. Intuitively, one expects to see some non-trivial behavior when i is of order θ. To see that set $i = u\theta$. Then K_θ transforms into

$$K_\theta(u\theta, u\theta) = \frac{1}{(2\pi\mathbf{i})^2} \oint \oint \exp\Big(\theta(S(v) - S(w))\Big) \frac{\sqrt{vw}}{v - w} \frac{dvdw}{vw}, \tag{2.5}$$

with

$$S(z) = z - z^{-1} - u \ln z.$$

Our next aim is to deform the contours of integration so that $\Re(S(v) - S(w)) < 0$ on them. (It is ok if $\Re(S(v) - S(w)) = 0$ at finitely many points.) If we manage to do that, then (2.5) would decay as $\theta \to \infty$. Let us try to do this. First, compute the critical points of $S(z)$, i.e. roots of its derivative

$$S'(z) = 1 + z^{-2} - uz^{-1}.$$

When $|u| < 2$ the equation $S'(z) = 0$ has two complex conjugated roots of absolute value 1 which we denote $e^{\pm i\phi}$. Here ϕ satisfies $2\cos(\phi) = u$. Let us deform the contours so that both of them pass through the critical points and look as shown at Figure 2. We claim that now $\Re S(v) < 0$ everywhere on

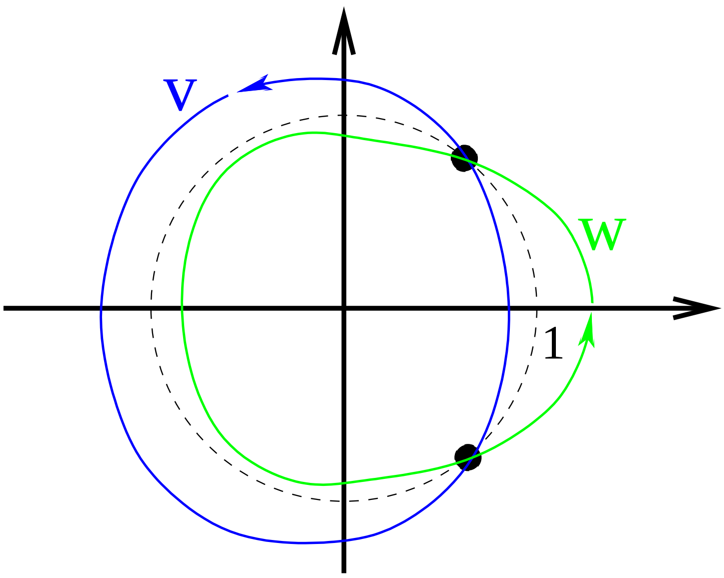

Figure 2: Deformed contours: v–contour in blue and w–contour in green. The dashed contour is the unit circle and the black dots indicate the critical points of $S(z)$.

its contour except at critical points $e^{\pm i\phi}$, and $\Re S(w) > 0$ everywhere on its contour except at critical points $e^{\pm i\phi}$ ($\Re S(v) = \Re S(w) = 0$ at $e^{\pm\phi}$.) To prove that observe that $\Re S(z) = 0$ for z on the unit circle $|z| = 1$ and compute the gradient of $\Re S(z) = \Re S(a + bi)$ on the unit circle (i.e. when $a^2 + b^2 = 1$):

$$\Re S(a+bi) = a - \frac{a}{a^2+b^2} - \frac{u}{2}\ln(a^2+b^2),$$

$$\nabla \Re S(a+bi) = \left(1 - \frac{b^2-a^2}{(a^2+b^2)^2} - \frac{au}{a^2+b^2}\,,\, \frac{2ab}{(a^2+b^2)^2} - \frac{bu}{a^2+b^2}\right)$$
$$= \big(1 - b^2 + a^2 - au\,, 2ab - bu\big) = \big(2a^2 - au\,, 2ab - bu\big) = (2a-u)(a,b). \tag{2.6}$$

Identity (2.6) implies that the gradient vanishes at points $e^{\pm i\phi}$, points outwards the unit circle on the right arc joining the critical points and points inwards on the left arc. This implies our inequalities for $\Re S(z)$ on the contours. (We assume that the contours are fairly close to the unit circle so that the gradient argument works.)

Now it follows that after the deformation of the contours the integral vanishes as $\theta \to \infty$. Does this mean that the correlation functions also vanish? Actually, no. The reason is that the integrand in (2.5) has a singularity at $v = w$. Therefore, when we deform the contours from the contour configuration with $|w| < |v|$, as we had in Corollary 2.8, to the contours of Figure 2 we get a residue of the integrand in (2.5) at $z = w$ along the arc of the unit circle joining $e^{\pm i\phi}$. This residue is

$$\frac{1}{2\pi \mathbf{i}} \int_{e^{-i\phi}}^{e^{i\phi}} \frac{dz}{z} = \frac{\phi}{\pi}.$$

We conclude that if $u = 2\cos(\phi)$ with $0 < \phi < \pi$, then

$$\lim_{\theta\to\infty} K_\theta(u\theta, u\theta) = \frac{\phi}{\pi}.$$

Turning to the original picture we see that the asymptotic density of particles at point i changes from 0 when $i \approx 2\theta$ to 1 when $i \approx -2\theta$. This means that after rescaling by the factor θ^{-1} times the Plancherel–random Young diagram asymptotically looks like in Figure 3. This is a manifestation of the Vershik–Kerov–Logan–Shepp limit shape theorem, see [Vershik-Kerov-77], [Logan-Shepp-77].

More generally, we have the following theorem.

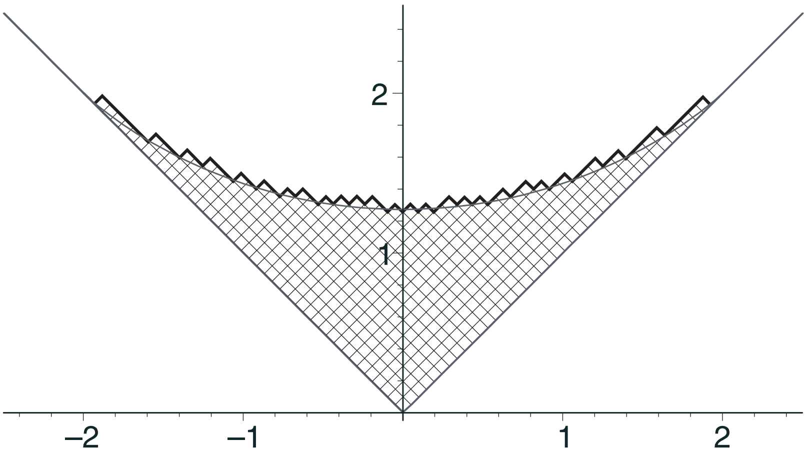

Figure 3: The celebrated Vershik–Kerov–Logan-Shepp curve as a limit shape for the Plancherel random Young diagrams.

Theorem 2.9 ([Borodin-Okounkov-Olshanski-00])**.** *For any* $-2 < u < 2$ *and any two integers* x, y *we have*

$$\lim_{\theta\to\infty} K_\theta(\lfloor u\theta\rfloor + x, \lfloor u\theta\rfloor + y) = \begin{cases} \dfrac{\sin(\phi(x-y))}{\pi(x-y)}, & \text{if } x \neq y, \\ \dfrac{\phi}{\pi}, & \text{otherwise,} \end{cases} \tag{2.7}$$

where $\phi = \arccos(u/2)$*.*

Remark. The right–hand side of (2.7) is known as the *discrete sine kernel* and it is similar to the *continuous sine kernel* which arises as a *universal* local limit of correlation functions for eigenvalues of random Hermitian (Wigner) matrices.

So far we got some understanding on what's happening in the *bulk*, while we started with the Last Passage Percolation which is related to the so-called *edge* asymptotic behavior, i.e. limit fluctuations of λ_1. This corresponds to having $u = 2$, at which point the above arguments no longer work. With some additional efforts one can prove the following theorem:

Theorem 2.10 ([Borodin-Okounkov-Olshanski-00],[Johansson-01a])**.** *For any two reals* x, y *we have*

$$\lim_{\theta\to\infty} \theta^{1/3} K_\theta(2\theta + x\theta^{1/3}, 2\theta + y\theta^{1/3}) = K_{Airy}(x, y) \tag{2.8}$$

where

$$K_{Airy}(x, y) = \frac{1}{(2\pi\mathbf{i})^2} \int\int e^{\widetilde{v}^3/3 - \widetilde{w}^3/3 + \widetilde{v}x - \widetilde{w}y} \frac{d\widetilde{v}d\widetilde{w}}{\widetilde{v} - \widetilde{w}}, \tag{2.9}$$

with contours shown at the right panel of Figure 4.

Remark 1. Theorem 2.10 means that the random point process $X(\lambda)$ "at the edge", after shifting by 2θ and rescaling by $\theta^{1/3}$, converges to a certain non-degenerate determinantal random process with state space $\mathbb{R}$ and correlation kernel K_{Airy}.

Remark 2. As we will see, Theorem 2.10 implies the following limit theorem for the first number of the partition λ, λ_1: For any $s \in \mathbb{R}$

$$\lim_{\theta\to\infty} \mathbb{P}(\lambda_1 \le 2\theta + s\theta^{1/3}) = \det(\mathbf{1} - K_{Airy}(x,y))_{L_2(s,+\infty)}.$$

One shows that the above *Fredholm determinant* is the Tracy–Widom distribution $F_2(s)$.

Proof of Theorem 2.10. We start as in the proof of Theorem 2.9. When $u = 2$ the two critical points of $S(z)$ merge, so that the contours now look as in Figure 4 (left panel) and the integral in (**??**) vanishes. Therefore, the correlation functions near the edge tend to 0. This is caused by the fact that points of our process near the edge rarify, distances between them become large, and the probability of finding a point in any given location tends to 0.

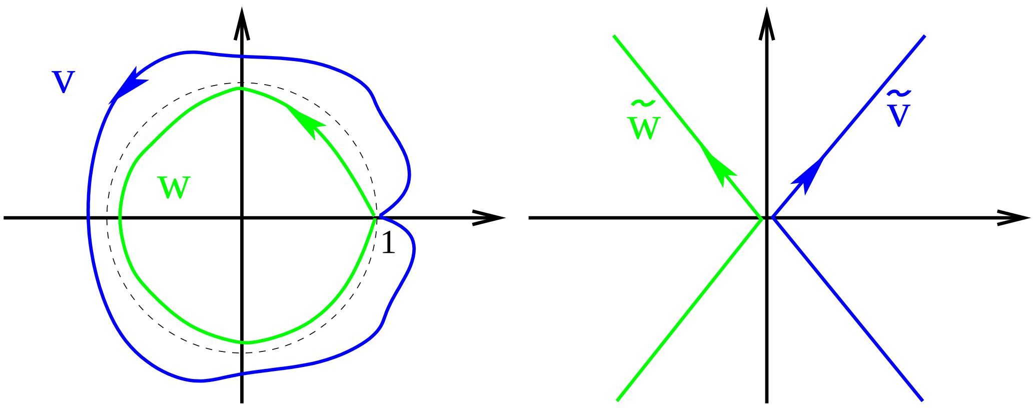

Figure 4: Contours for the edge–scaling limit ($u = 2$). Left panel: v–contour in blue and w–contour in green. The dashed contour is the unit circle. Right panel: limiting contours.

In order to see some nontrivial behavior we need *rescaling*. Set

$$v = 1 + \theta^{-1/3}\widetilde{v}, \quad w = 1 + \theta^{-1/3}\widetilde{w}$$

in the contour integral. Note that $z = 1$ is a double critical point of $S(z) = z - z^{-1} - 2\ln(z)$, so that in the neighborhood of 1 we have

$$S(z) = \frac{1}{3}(z-1)^3 + O((z-1)^4)$$

Now as $\theta \to \infty$ we have

$$\exp\Big(\theta(S(v)-S(w))\Big) = \exp\left(\theta\left(\frac{1}{3}(\theta^{-1/3}\widetilde{v})^3 - \frac{1}{3}(\theta^{-1/3}\tilde{w})^3\right) + o(1)\right)$$
$$= \exp\left(\frac{1}{3}\widetilde{v}^3 - \frac{1}{3}\widetilde{w}^3\right).$$

We conclude that as $\theta \to \infty$

$$K_\theta(2\theta + x\theta^{1/3}, 2\theta + y\theta^{1/3}) \approx \frac{\theta^{-1/3}}{(2\pi\mathbf{i})^2}\int\int e^{\widetilde{v}^3/3-\widetilde{w}^3/3+\widetilde{v}x-\widetilde{w}y}\frac{d\widetilde{v}d\widetilde{w}}{\tilde{v}-\tilde{w}}, \qquad (2.10)$$

and the contours here are contours of Figure 4 (left panel) in the neighborhood of 1; they are shown at the right panel of Figure 4. □

3 Philosophy behind

The conceptual conclusion from all the above is that for this random line model which has rich algebraic structures, we can actually express it as a determinantal process. By symmetric function theory we can compute the correlation kernel explicitly as double contour integral. Having this, many limiting questions can be answered by analyzing these integrals. And processes related to random matrices arise naturally around non-trivial points. The method for the analysis that we presented is, actually, quite standard and is well-known (at least since the XIX century) under the *steepest descent method* name. In the context of determinantal point processes and Plancherel measures it was introduced by Okounkov.

To be precisely, by computing the one point correlation function, we identify the limit shape of random curve after suitable scaling, since the one point correlation function is the slope of the curve. It turns out that the limit shape it has is frozen outside a interval$(-2, 2)$. This interval corresponding to the regime that the steepest decent method is trivial. We call points in the interval $(-2, 2)$ bulk points. Then for each bulk point the critical point is or order 2. In a window of suitable size, we get discrete sine process. For edge point $-2, 2$, the critical points are of order 3, in a window of suitable size, we observe Airy process.

4 Random surface story

If we consider random step surface instead of random curve, we have the similar story. Here we only describe the corresponding methodology and phenomena.

We can associate a two dimensional partition π with a 3D Young diagram, which determine a surface. Weight the 3D Young diagram by Boltzmann weight according to volume, $q^{\mathrm{Vol}(\pi)}$. We get a random surface model. We are interest in the behaviour of this surface when $q \to 1$.

Okounkov and Reshetikhin[Okounkov-Reshetikhin-01] construct two dimensional extension of Schur measure, which is Schur process to study this model. And like Poissonized Planchered measure can be interpreted as a Schur process. By the theory of symmetric function, Schur process is a two dimensional determinantal process whose correlation kernel is also given by a double contour integral. We can also carry the steepest decent method.

By take the limit of the 1-point correlation function. One find a trivial region with is constant and a non-trivial region. We can identify the deterministic limit shape of the random surface by integrate against the 1point correlation function The trivial region is called ordered and the non-trivial region is called disordered.The boundary between this two regions is called frozen boundary. We observe different process around different types of non trivial points by steepest decent method.

1. In the bulk(2nd order singularity) of the disordered the correlation are given by the incomplete beta kernel with the parameters determined by slope of the limit shape (a special case of this is the discrete sine kernel).

2. At a general point of the frozen boundary(3rd order singularity), suitably scaled, the correlation are given by the extended Airy kernel, which also describe top line of Dyson Brownian motion(Airy_2 process[**?**]).

3. At a cusp(4th order singularity) of the frozen boundary, correlation, suitably scaled, are given by the extended Pearcey kernel, which appears in a random matrix model and a Brownian motion model for a fixed time[TW-06].

4. At a turning point, a point of the frozen boundary where the disordered region meets two different frozen phases, we observe the GUE corner process.

5. At a cuspidal turning point, which is a cusp and turning point at the same time, we observe a deformed Airy process.

6. For unbounded domains, the so-called tentacles, which are the exponentially narrow channels of disorder separating unbounded frozen regions, the local statistics inside a tentacle is called bead model(see [Boutillier-09]). It can be seen as a two side extension of GUE corner

process. In addition, we point out that this process is not obtained by Schur process mechanism.

For the proof of these results, see [BG12], [Okounkov-Reshetikhin-01], [Okounkov-Reshetikhin-05],[OR06] [Boutillier-09].

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